7.1 Introduction

To study structures, such as current sheets, flux ropes, or fully three-dimensional configurations, it is desirable to examine them in their proper (co-moving) frame, in which they appear as time stationary as the data permit. And it is important to establish the frame velocity so that their physical dimensions can be determined. In the present chapter, we discuss methods for finding the velocity of the proper frame, mostly from single spacecraft data. The so-called Walén test is also discussed in the chapter. Its purpose is to identify one-dimensional Alfvénic structures from single-spacecraft data, usually in the context of magnetic field reconnection geometries or interplanetary discontinuities.

The simplest situation is one where the electric field in the proper frame is negligibly small. When such is the case, the co-moving frame is called the deHoffmann-Teller (HT) frame. It was first applied by de Hoffmann and Teller [1950] in a theoretical study of the one-dimensional structure of MHD shocks. In this application, the HT frame was specified by the requirement that the electric field on both sides of the shock, but not necessarily in the middle of it, was zero. In other words, the plasma flow on the two sides was field-aligned, when viewed in the HT frame. The component of the frame velocity, $V_{HT}$, along the direction normal to a one-dimensional layer represents the motion of the layer, while the tangential component represents what has been called the field-line velocity. Important applications of the HT frame include the study of particle reflection and acceleration at shocks (see Section 8.2.1), at the magnetopause, and in the geomagnetic tail current sheet. It is also used in the Grad-Shafranov reconstruction methods presented in Chapter 9.

The concept of an HT frame is not limited to one-dimensional structures. Such a frame can exist for some two- and three-dimensional objects as well. But there are also structures which possess an intrinsic electrostatic field that cannot be transformed away. An example is the standard two-dimensional reconnection configuration, which, in its proper frame (the frame moving with the X-line), has a remnant electric field along the invariant (axial) direction. Determination of the magnitude of this reconnection-generated field is an important goal. To reach it, a high-quality determination of the proper frame is required.
The procedure, first developed in Sonnerup et al. [1987], for obtaining the HT frame velocity and, if so desired, its acceleration, from data measured by a single spacecraft was described in Chapter 9 of ISSI SR-001 [Khrabrov and Sonnerup, 1998]. Comments on the procedure and applications are given in Section 7.2 below. The more general situations, in which no HT frame exists, i.e., where the structures have an intrinsic remaining electrostatic field in their proper frame are discussed in Section 7.3.

7.2 The deHoffmann-Teller frame

As shown in Chapter 9 of ISSI SR-001, the procedure for determining a HT frame consists of finding the transformation velocity, $V_{\text{HT}}$, from the spacecraft frame to the HT frame that minimises the residual electric field in the least-squares sense. This is accomplished by minimising the object function

$$D(V) = \langle |E'|^2 \rangle = \langle |E + V \times B|^2 \rangle$$

(7.1)

with respect to the frame velocity, $V$. Here the angle brackets $\langle \ldots \rangle$ denote the average over the data set used in the calculation. The minimum corresponds to $V = V_{\text{HT}}$. In other words, $V_{\text{HT}}$ is determined from least-squares fitting of $-V_{\text{HT}} \times B_i$ to the measured electric field vectors $E_i$. The components of $V_{\text{HT}}$ satisfy the equation $\langle E' \times B \rangle = 0$, where the electric field in the co-moving frame is $E' = E + V \times B$. In matrix form, the equations for the frame velocity components become

$$\begin{pmatrix}
\langle B_y^2 + B_z^2 \rangle & \langle -B_x B_y \rangle & \langle -B_x B_z \rangle \\
\langle -B_x B_y \rangle & \langle B_x^2 + B_z^2 \rangle & \langle -B_y B_z \rangle \\
\langle -B_x B_z \rangle & \langle -B_y B_z \rangle & \langle B_x^2 + B_y^2 \rangle
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}
= \begin{pmatrix}
\langle E_y B_z - E_z B_y \rangle \\
\langle E_z B_x - E_x B_z \rangle \\
\langle E_x B_y - E_y B_x \rangle
\end{pmatrix}$$

(7.2)

The uncertainties in the $V_{\text{HT}}$ determination are discussed in detail in Chapter 9 of ISSI SR-001.

If only the electric field components in the spacecraft spin plane are measured, but not the spin-axis component, one commonly used procedure is to calculate the spin-axis component by use of the assumption $E_i \cdot B_i = 0$, although this procedure is not always physically justified and also gives problems when $B$ lies near the spin plane. Another option is described in Section 7.3.1.

In case no direct electric field measurements are available, but plasma velocities are, the $E_i$ vectors are replaced by $-v_i \times B_i$. Using $-v_i \times B_i$ as proxy for the electric field has been the standard procedure in the past, commonly based on ion bulk velocities.

If reliable electron bulk velocities are available, they offer a considerable advantage, as pointed out in Chapter 9 of ISSI SR-001 and by Puhl-Quinn and Scudder [2000]. Ion and electron bulk velocities will differ as soon as there are significant electric currents, $j = ne(v_i - v_e)$. In this case, the use of the electron velocities is the better choice, because the magnetic field lines are more closely tied to the electron fluid. In terms of the generalised Ohm’s law, it implies that the Hall electric field, which can be significant, is incorporated in the analysis.

As discussed in Section 9.2 of ISSI SR-001, the HT frame participates in the motion of a boundary or other discontinuity. Thus $V_{\text{HT}} \cdot \hat{n}$ represents the velocity of motion of the discontinuity along its normal, assuming a good normal vector is known (see also...
Section 10.5.5 [Schwartz, 1998] of ISSI SR-001). In Chapter 1 of the present volume, methods for determining normals are discussed, and so are other methods to determine discontinuity speeds from single spacecraft measurements, such as the Minimum Faraday Residue (MFR) and Minimum Mass-flux Residue (MMR) methods.

7.3 General proper frame

Here we discuss the more general situation where no ideal HT frame exists, but where a proper frame (in which the structure is time independent but has an intrinsic electrostatic field distribution) does exist: it is this frame that one wants to find.

7.3.1 One-dimensional structures

In one-dimensional structures such as shocks, rotational and tangential discontinuities there can be, and probably often is, an intrinsic electric field, $E_n$, along the normal within the layer itself. The presence of such an electric field can adversely influence the quality of the determination of the proper velocity from the standard HT algorithm. And this intrinsic field, with its associated electric potential, is itself of importance in understanding some of the physical processes operating in the layer. Therefore it is desirable to properly incorporate the intrinsic field in the analysis. Perhaps the simplest way to achieve this goal is to manually exclude data points within the layer and simply apply the standard HT algorithm to electric field data or to $-v \times B$ data taken on the two sides. After $V_{HT}$ has been found in this manner, transforming the electric field within the layer to the HT frame should produce the desired intrinsic field and potential. Ideally this electric field should come from direct three-dimensional field measurements. By then examining the portions contributed separately by $-v \times B$ and by $-v_e \times B$, one can assess the individual contributions in Ohm’s law from the convection electric field, from the Hall field, and from the electron pressure gradient.

An alternate approach (as yet untested) is to transform the measured electric field near and within the layer to a frame of reference in which only the tangential part of the electric field is minimised. The appropriate object function for minimisation with respect to the three components of the frame velocity, $V$, is then

$$D(V) = \langle |\dot{n} \times (E + V \times B)|^2 \rangle$$  \hfill (7.3)

Here $\dot{n}$ is the normal vector, assumed known, and, as before, the angle brackets $\langle \ldots \rangle$ denote the average over the data set. Using a right-handed coordinate system, defined by a set of unit vectors $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$, with $\hat{n} = \hat{x}_3$, the minimisation leads to the following matrix equation for the velocity components:

$$\begin{pmatrix}
B_3
0
-B_1 B_3
0
B_3
-B_2 B_3
-B_1 B_3
-B_2 B_3
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
E_2 B_3 \\
-E_1 B_3 \\
E_1 B_2 - E_2 B_1 \\
\end{pmatrix}
$$  \hfill (7.4)

In the frame of reference moving with velocity $V$, the electric field should then be zero, or close to zero, except for an intrinsic electric field within the layer, directed along $\dot{n}$. This field can then be integrated to give the electric potential distribution and net potential jump.
The two tangential velocity components, $V_1$ and $V_2$, become undefined for a perfect tangential discontinuity (TD), i.e., for $B_3 = 0$. For this case, Eqn. 7.4 then gives $V_3 = (E_1 B_2 - E_2 B_1) / (B_1^2 + B_2^2)$. Regardless of the values of $V_1$ and $V_2$, the electric field remaining in any frame that has its velocity component along the normal equal to this value of $V_3$, is directed purely along $\hat{n}$, but is generally not zero outside the discontinuity layer. However, the two velocity components $V_1$ and $V_2$ can be determined in a separate step, as described by Paschmann [1985], to give zero electric field on the two sides of the discontinuity layer. After transformation to the frame of reference moving in this manner, only the intrinsic electric field within the layer remains, directed along $\hat{n}$. The velocity $V_3$ replaces $V_{HT}$ as the estimate of the current sheet motion along the normal.

As pointed out to us by A. Vaivads [private communication, 2007], there is also an entirely different application of Eqn. 7.4, namely, to the determination of a proper frame velocity $V$, from electric field components, $E_1$ and $E_2$, measured in the spin plane of a spacecraft that lacks the ability to measure the spin-axis component, $E_3$. Eqn. 7.4 can be derived from the corresponding formula for regular HT analysis (Eqn. 7.2 above; see also Eqn. 9.12 in ISSI SR-001, with $K_0$ defined as the average of the matrix $K_{\mu \nu}^{(m)}$ in Eqn. 9.11) by letting $\langle E_3 B_3 \rangle = \langle E'_3 B_3 \rangle = 0$, where $E' = E + V \times B$ is the electric field in the co-moving frame. These two conditions are not exactly satisfied in the HT frame, obtained from Eqn. 7.2. In the latter frame, one has instead that $\langle E'_3 B_3 \rangle = \langle E'_y B_z \rangle$ and $\langle E'_3 B_3 \rangle = \langle E'_x B_z \rangle$. Therefore, the frame velocity derived from the spin-plane application of Eqn. 7.4, is not identical to the regular HT velocity, except if $E' = 0$, as it is in a perfect HT frame, or if the four above correlations between components of $E'$ and $B$ happen to vanish. However, if a good HT frame does exist, the difference between $V_{HT}$ and the frame velocity $V$ from Eqn. 7.4 is expected to be small. The procedure again fails if $\langle B_z^2 \rangle$ is close to zero.

### 7.3.2 Two-dimensional structures

Two-dimensional, time-independent but moving structures in a plasma have the property (from Faraday’s law) that, in their proper frame, the electric field along the invariant direction is constant throughout the structure. This property can in some cases be used to determine both the direction and the motion of the invariant axis from single-spacecraft data. This possibility was mentioned in Chapter 9 of ISSI SR-001 but no convenient method for actually obtaining the orientation and motion was available at the time. A least-squares method for this purpose has now been developed [Sonnerup and Hasegawa, 2005] and, to a limited extent, tested with actual spacecraft data. In brief, the method again leads to an eigenvalue problem, in which the sought-after invariant axis, $k$, is predicted as the eigenvector $k_3$ corresponding to the smallest eigenvalue, $\lambda_3$, of the system

$$ (M_0 - \lambda I) \cdot k = 0 \quad (7.5) $$

where $I$ is the identity matrix and the symmetric $3 \times 3$ matrix $M_0$ is defined by

$$ M_0 = -M^{EB} \cdot M^{-BB} \cdot M^{BE} + M^{EE} \quad (7.6) $$

Here the matrices $M^{EB}$, $M^{BB}$, $M^{BE}$, and $M^{EE}$, are co-variance matrices, e.g., $M^{EB}_{ij} = \langle \delta E_i \delta B_j \rangle$, and $M^{-BB}$ is the inverse of the magnetic co-variance matrix $M^{BB}_{ij} = \langle \delta B_i \delta B_j \rangle$. 


As before, the symbol $\delta$ denotes deviation from the average, e.g., $\delta B_i = B_i - \langle B_i \rangle$ and $\langle \ldots \rangle$ denotes the average over the data set used in the calculation. Assuming $k$ to be normalised, $|k|^2 = 1$, the velocity of motion of the invariant axis is given by

$$V_0 = k \times (M^{-BB} \cdot M^{BE} \cdot k) \quad (7.7)$$

and the constant electric-field component along the invariant axis, evaluated in the system moving with velocity $V_0$, is

$$E_0 = \langle E \rangle \cdot k + \langle B \rangle \cdot (k \times V_0) \quad (7.8)$$

The method fails if $M^{BB}$ is not invertible and if a perfect deHoffmann-Teller (HT) frame exists. In a perfect HT frame, the electric field is identically zero and therefore cannot contain any information about axis orientation. To date, applications of the method presented above are limited but indicate that the technique can work well for some flux transfer events at Earth’s magnetopause [Sonnerup and Hasegawa, 2005; Hasegawa et al., 2006]. For ordinary magnetopause traversals it seldom gives believable results, usually because of poor separation of the two smallest eigenvalues of $M_0$.

Various multi-spacecraft methods for finding axis orientation (and motion) have been developed [Shi et al., 2005, 2006; Xiao et al., 2004; Zhou et al., 2006a, b] and applied to Cluster data (see also Chapter 2).

### 7.3.3 Three-dimensional structures.

Except for the special case where an acceptably good HT frame exists, there appears to be no obvious way to find the motion of a fully three-dimensional, time-independent structure from single-spacecraft data. However, if the Cluster spacecraft separation is sufficiently small compared to the size of the structure, one may use the gradient-based method developed by Shi et al. [2006] on the measured magnetic field. An alternate, as yet untested, approach is to use Cluster’s curlometer capability (see Chapter 2) to search for a frame of reference moving at velocity $V$ such that the object function

$$D(V) = \left| \nabla \times (E + V \times B) \right|^2 \quad (7.9)$$

obtained from Faraday’s law, written in the co-moving frame where $\partial B/\partial t = 0$, is minimised. If a perfect proper frame exists, the object function will be exactly zero. The resulting set of linear equations for the components of $V$ is lengthy and is not given here. As is the case for the gradient-based method [Shi et al., 2006], the Faraday-based approach would be applicable, not only to three-dimensional structures, but to one- and two-dimensional ones as well.

### 7.4 Walén relation

As plasma flows across an ideal rotational discontinuity (RD), the components of the plasma velocity, $v$, tangential to that layer change in response to the $j \times B_n$ force ($B_n = B \cdot \hat{n}$). In the spacecraft frame, this implies that

$$\Delta v = \pm \Delta v_A \quad (7.10)$$
where the symbol $\Delta$ refers to changes relative to some upstream state, for example, and $v_A$ is the local Alfvén velocity, corrected for the effect of pressure anisotropy,

$$v_A = B\left[(1 - \alpha)/\mu_0\rho\right]^{0.5} \tag{7.11}$$

with $\alpha = (p_\parallel - p_\perp)\mu_0/B^2$. Eqn. 7.10 is the so-called Walén relation expressed in the spacecraft frame.

In spite of the fact that the flow acceleration $\Delta v$ is due to the $\mathbf{j} \times \mathbf{B}$ force, the Walén relation does not contain $B_n$. This is because for an RD, as for any planar Alfvén wave, $v_n$ is proportional to $B_n$. Thus the smaller $B_n$, the smaller the inflow velocity, and thus the less mass to accelerate. Of course, this independence of $B_n$ means that a successful Walén test does not say anything about the reconnection rate, which is proportional to $B_n$. On the other hand, it does imply that $B_n$ was non-zero because otherwise the $\mathbf{j} \times \mathbf{B}$ force would be zero, i.e., there would be no tangential acceleration of the plasma. This is no small feat because the expected $B_n$ is so small, compared with the total $B$, that to experimentally determine that its value, $\mathbf{B} \cdot \hat{n}$, is significantly different from zero requires a very nearly one-dimensional structure and knowledge of the direction, $\hat{n}$, normal to it with an accuracy rarely available (see Chapter 1).

The positive (negative) sign in the Walén relation applies if the normal components of the magnetic field and plasma velocity, $B_n$ and $v_n$ have the same (opposite) signs.

When formulated as a jump relation, the test requires selection of the times between which the jumps in $v$ and $v_A$ are to be compared. This problem is partially overcome if, after selection of an upstream reference state, one produces a time-series plot, where the velocities predicted by the Walén relation are overplotted on the observed velocities [e.g., Phan et al., 2004]. Since there is the issue of the a-priory unknown sign in the Walén relation, one can overplot the predicted velocities, once with the $+$ sign and a second time with the $-$ sign. Such time-series plots have the advantage of showing, for example, whether the relation is fulfilled during only part of the discontinuity crossing.

As discussed in Section 9.3.3 of ISSI SR-001, the HT frame provides a convenient frame for testing the Walén relation. In the HT frame, the Walén relation has a much simpler formulation. As the plasma velocity in this frame becomes $v' = (v - V_{HT})$, the Walén relation reduces to

$$v' = \pm v_A \tag{7.12}$$

In the HT frame, testing of the Walén relation thus consists of a component-by-component scatter plot of the plasma velocities and the Alfvén velocities measured across the discontinuity. There is no need for selecting reference values for $v$ and $v_A$. When applied to an ideal RD, the scatter plot should show a correlation coefficient and a slope of $\pm 1$. A scatter plot destroys the time-order of the measurements. If one wants to preserve this order, one can alternatively overplot $(v - V_{HT})$ and $\pm v_A$.

If the Walén test is used on a slow-mode shock, the flow speed and Alfvén speed are no longer equal: the upstream Alfvén number is larger than the downstream value and the former is $<1$, except for a switch-off shock, for which it is equal to 1. On the other hand, an intermediate shock has upstream Alfvén number $>1$ while its downstream Alfvén number remains $<1$. If the test is used on a reconnection configuration, consisting of an exhaust jet sandwiched between two slow-mode shocks, such as might occur in the geomagnetic tail, the scatter plot should show two branches, one with positive and one with negative correlation. This behaviour would unambiguously indicate plasma inflow from both sides.
of the wedge. This result is more convincing than a direct determination of the normal flow by projection of the flow vectors in the HT frame onto the normal vector. The direction of the latter would have to be obtained by use of one of the methods described in Chapter 1 and such normals have notoriously large uncertainties.

A final remark is that the existence of an HT frame is a necessary but not a sufficient condition for the identification of a discontinuity as a one-dimensional RD. This statement holds even when data points within the layer are included in the HT analysis. For example, one can imagine a perfect tangential discontinuity (TD) where no intrinsic electric field along the normal direction is present within the discontinuity. Such a TD would possess a perfect HT frame but would not be an RD.

### 7.5 Recent applications

Use of the HT frame has become widespread, either as the frame in which moving structures are conveniently discussed [Hasegawa et al., 2004, 2005, 2006; Sonnerup et al., 2004; Sonnerup and Hasegawa, 2005; Nykyri et al., 2006], or to estimate the speed of a discontinuity as the normal component of the transformation velocity, $V_{HT} \cdot \mathbf{\hat{n}}$ [Haaland et al., 2004; Khotyaintsev et al., 2004; Sonnerup et al., 2006, 2007; Retinò et al., 2006], or as the convenient frame in which to test the Walén-relation [Puhl-Quinn and Scudder, 2000; Eriksson et al., 2004a, b; Phan et al., 2004; Paschmann et al., 2005; Gosling et al., 2005; Nykyri et al., 2006].

In all the cited applications, the HT frame determination was based on the measured ion bulk velocities and magnetic fields, with the exception of Khotyaintsev et al. [2004], where the measured electric field was used directly. Electric field data have been used only rarely because the Cluster electric field measurements are only two-dimensional. To our knowledge, the only paper where electron bulk velocities were used is Puhl-Quinn and Scudder [2000].

### 7.6 Summary

In this chapter, the topics of the deHoffmann-Teller (HT) frame and the Walén relation from Chapter 9 of ISSI SR-001 have been revisited. In particular, the HT frame is now viewed as a special case of the more general concept of a ‘proper frame’, i.e., a frame co-moving with a plasma/field structure, in which frame the structure is assumed to be approximately time independent but may possess an intrinsic remnant electrostatic field distribution. The structure studied may be one-, two-, or three-dimensional and the HT frame corresponds to the case where the intrinsic electric field is zero, or negligibly small. In using these tools, it is important to keep in mind some of the following pitfalls and unexplained features.

The velocity of a one-dimensional discontinuity along the normal direction is, in principle, given by $V_{HT} \cdot \mathbf{\hat{n}}$. But in many applications, e.g., at the magnetopause, $V_{HT}$ is a large velocity that is nearly perpendicular to the normal vector $\mathbf{\hat{n}}$, the result being that the usually much smaller normal velocity component is very sensitive to errors both in frame velocity and in normal direction. These errors are not only of a statistical nature. They may have large contributions from geometrical substructures and from a lack of time stationar-
ity. Limited insight into these uncertainties can be gained by repeated determinations of $V_{HT}$ and $\hat{n}$, using nested data segments. For two-dimensional structures where a reliable invariant direction has been found, a more reliable estimate of the normal motion may be $V_{HT,\perp} \cdot \hat{n}$, where $V_{HT,\perp}$ is the portion of the HT velocity perpendicular to the invariant axis.

One persistent problem with the Walén test, encountered in its application at the magnetopause, is that plasma mass densities, $\rho$ and pressure anisotropies, $\alpha$, measured at, and near, the magnetopause, do not follow the MHD-based theoretical prediction for a rotational discontinuity, namely that $\rho(1 - \alpha)$ should remain constant within the layer. The reason for this discrepancy is not understood. Inclusion of data from magnetospheric regions, in which the plasma has not yet interacted with the magnetopause, could be a contributing factor. In such regions the plasma velocity is small and the Alfvén speed large. It has been found that the quality of the Walén test usually improves, if, by use of the above MHD-based prediction, one replaces the measured plasma density within the layer by $\rho = \rho_1(1 - \alpha_1)/(1 - \alpha)$, where the subscript one denotes the upstream condition [Paschmann et al., 1986; Phan et al., 2004; Retinò et al., 2005].

In applications to the magnetopause [e.g., Paschmann et al., 2005] and also to solar-wind discontinuities [e.g., Neugebauer, 2006], the directional changes of the plasma velocity are often found to be in good agreement with the corresponding changes of the Alfvén velocity, while the magnitude of the velocity changes are usually considerably smaller than those of the Alfvén speed. The agreement of the directional changes is consistent with the presence of magnetic connection across the layer; the disagreement of the magnitude changes indicates that some unknown contributions to the tangential stress balance are present but not accounted for.

There are several possible explanations for the noted discrepancies in the Walén test. For example, cold particle populations or heavier ions may have been missed in the measurements so that the actual Alfvén speed is smaller than that used in the test. The presence of gradients tangential to the discontinuity surface of plasma pressure and guide field may also change the tangential stress balance so that the purely one-dimensional structure assumed in the Walén test becomes invalid. If the reconnection jets are confined to a narrow longitude segment, i.e., if the X-line is short, the plasma in the jets may be magnetically coupled to adjoining regions where no reconnection occurs. The presence of such three-dimensional effects cannot be excluded. Another possibility is that, in some crossings, the Walén relation may be rendered invalid in the interior of the magnetopause by non-gyroscopic plasma behaviour such as the presence of some form of gyro-viscosity [e.g., Hau and Sonnerup, 1991].

Note that incident, transmitted, reflected, and trapped particle populations, all contribute to the tangential stress balance and therefore should be included in the Walén test, their effects being accounted for via their contribution to both the total density and the pressure anisotropy factor. It has been reported by Retinò et al. [2005] that the quality of the Walén test can be improved significantly by removing a beam-like, counter-streaming, secondary particle population that is sometimes seen within (but not outside) the magnetopause current layer, in addition to the transmitted beam. The reasons for the presence of this secondary beam, or for not including it in the tangential stress balance, are not understood at present.
Software implementation

Software for the determination of the HT frame and for testing the Walén relation is included in QSAS, available at http://www.sp.ph.ic.ac.uk/csc-web/QSAS/.

Acknowledgements

The work by B.U.Ö.S. was supported by the National Aeronautics and Space Administration under Cluster Theory Guest Investigator Grant NNG-05-GG26G to Dartmouth College.

Bibliography


