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Weak Magnetohydrodynamic Turbulence of a Magnetized Plasma

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Abstract—A weak turbulence of the magnetohydrodynamic waves in a strongly magnetized plasma was studied in the case when the plasma pressure is small as compared to the magnetic field pressure. In this case, the principal nonlinear mechanism is the resonance scattering of fast magnetoacoustic and Alfvén waves on slow magnetoacoustic waves. Since the former waves are high-frequency (HF) with respect to the latter, the total number of HF waves in the system is conserved (adiabatic invariant). In the weak turbulence regime, this integral of motion generates a Kolmogorov spectrum with a constant flux of the number of HF waves toward the longwave region. The shortwave region features a Kolmogorov spectrum with a constant energy flux. An exact angular dependence of the turbulence spectra is determined for the wave propagation angles close to the average magnetic field direction. *© 2001 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The central place in the theory of turbulence belongs to the concept of a turbulence spectrum representing the energy distribution over scales. Determining the turbulence spectrum is a difficult problem that still remains unsolved. Important results in this field were obtained by Kolmogorov [1] and Obukhov [2], which showed an automodel character of the spectrum of developed hydrodynamic turbulence.

In the 1970s, the ideas of Kolmogorov and Obukhov were fruitfully developed and applied, mostly due to the effort of Zakharov, in the theory of weak wave turbulence (see monograph [3] and the first original papers [4–6]). The wave turbulence has proved to be, in a certain sense, somewhat simpler than the hydrodynamic turbulence. The presence of a wave dispersion results in that there exists a wave intensity region where the interaction between waves can be considered as weak. If the initial phase distribution of the waves is random, the weak nonlinear interaction provides for a small correlation between phases of the interacting waves. For this reason, the waves can be described in terms of the pair correlation functions with the Fourier images coinciding (to within a factor) with the number of waves n_k (occupation number) possessing a given wavevector $\mathbf{k}$. In turn, the occupation numbers n_k obey the kinetic wave equations. In this theory, the Kolmogorov spectra appear in the form of stationary scale-invariant solutions to the kinetic equations, corresponding to zeros of the collisional term. These spectra, in contrast to the thermodynamically equilibrium ones, refer to solutions of the flux type realizing a constant flux of some integral of motion (energy, number of particles, etc.) over scales. It is important to note that the concept of the interval of inertia (a region where the pumping and damping effects can be ignored), which is formulated as an assumption (a hypothesis of the locality of interaction) in the case of a developed hydrodynamic turbulence, is explicitly established as the locality of spectra for the weak wave turbulence.

Most of the investigations devoted to the Kolmogorov spectra of weak turbulence refer to isotropic media (for a complete bibliography, see [3]). The effect of anisotropy, for example, of the magnetic field in a plasma, was studied to a smaller extent. The first example of determining the Kolmogorov spectra in anisotropic media for a weak turbulence of magnetized ion-sound waves was reported by the author in 1972 [7]. It was found that the collisional term in the kinetic wave equations is invariant with respect to stretching in the two independent directions (along and across the magnetic field), which allowed the anisotropic Kolmogorov spectra to be constructed with a power dependence on both longitudinal (k_z) and transverse ($k_\perp$) components of the wavevector. This, in turn, made it possible to determine (with the aid of generalized Zakharov transformations) the Kolmogorov indices and find the exact angular dependence of the Kolmogorov spectra. Later, the ideas of that study were used in determining the turbulence spectra of the drift waves and the Rossby waves (see, e.g., [8, 9]).

This paper is devoted to the study of a weak turbulence of the magnetohydrodynamic (MHD) waves in a strongly magnetized plasma in the case when the plasma (thermal) pressure nT is small as compared to the magnetic field pressure $H^2/8\pi$:

$$\beta = \frac{8\pi nT}{H^2} \ll 1.$$

Under these conditions, the turbulence spectra are determined (unlike the cases studied previously [4, 7]) by solving three interrelated kinetic equations for the Alfvén waves and the fast and slow magnetoacoustic waves.

For $\beta \ll 1$, the main nonlinear interaction of MHD waves is the scattering of fast magnetoacoustic and Alfvén waves on slow magnetoacoustic waves (Section 2). In these processes (involving the decay of one wave into two, as well as the reverse process of merging), Since the fast magnetoacoustic and Alfvén waves act as high-frequency (HF) with respect to the slow magnetoacoustic waves. In every scattering event, a change in the frequency of the former waves (referred to below as the A-waves) is relatively small (due to the small β value), which makes this process analogous to the Mandel'shtam–Brillouin scattering of electromagnetic waves on acoustic phonons. As a result of this time scale separation, whereby the waves are divided into HF and low-frequency (LF) components, the wave-decay interaction retains, in addition to the energy, an adiabatic invariant—the total number of HF waves. This, however, does not exhaust the analogy with the Mandel'shtam–Brillouin scattering. It is established that the matrix element of this interaction is maximum for a maximum value of the longitudinal momentum component transferred from A-waves to slow magnetoacoustic waves. This result can be derived, in particular, from an expression derived by Galeev and Oraevskii [10] for the increment of the decay instability of a monochromatic Alfvén wave. It should be recalled that the matrix element for the Mandel'shtam–Brillouin scattering is proportional to the square root of the transmitted momentum, which accounts for the maximum backscattering of electromagnetic waves. In view of this behavior of the A-wave scattering amplitude, it is naturally assumed that a stationary angular distribution of these waves must be strongly anisotropic and concentrated along the magnetic field direction in the k-space. Under these assumptions, the kinetic equations acquire additional symmetry and become invariant with respect to stretching in the two independent directions (along and across the magnetic field), which allows the transformations developed previously [7] to be used in tis case as well.

Owing to these two features of the kinetic equations in the transparency range, it was possible to find two scale-invariant (with respect to longitudinal and transverse wavevectors) Kolmogorov spectra corresponding to a constant energy flux directed toward the shortwave region of scales (forward cascade) and a constant flux of the number of A-waves toward the region of small k (reverse cascade). This study is based on the results reported long ago in the form of a preprint in Russian [11] and remained, for this reason, unavailable abroad. Moreover, it turned out that the work was also little known in Russia: despite an almost three-decade history, the results are still not repeated. Recently, however, the question of MHD turbulence spectra was studied in the other limiting case ($\beta \gg 1$) [12]. This limit significantly differs from that considered below. First, a plasma with $\beta \gg 1$ can be treated as incompressible liquid. Second, this limit introduces no significant difference between the Alfvén waves and slow magnetoacoustic waves: the waves of both types exhibit the same dispersion law and differ only by polarization. Such a degeneracy significantly changes the character of nonlinear interactions. Nevertheless, this case also admits two types of the Kolmogorov spectra featuring dependence on the wavenumber analogous to that reported below. However, a physical explanation of the two spectra existing in the case of $\beta \gg 1$ is different from the interpretation given below for $\beta \ll 1$.

The material is arranged as follows. Section 2 provides for a canonical description of the ideal MHD wave turbulence following the original work of Zakharov and the author [13] and the recent review [14]. Using the Hamiltonian approach, Section 3 derives averaged equations describing the interaction of A-waves with slow magnetoacoustic waves. It is shown that the A-waves represent an HF force acting upon the slow magnetoacoustic waves. Since the potential of this force is negative (in contrast to the potential of interaction between the Langmuir waves and the ion-sound waves), the plasma is drawn into the regions of A-wave localization to form the density "humps." Stability of a monochromatic A-wave is also studied in Section 3. Section 4 describes the Kolmogorov spectra of a weak MHD turbulence.

2. VARIATION PRINCIPLE AND NORMAL VARIABLES

Let us consider the ideal MHD equations for a barotropic flow in a plasma, the internal energy ε of which can be considered as dependent only on the plasma density ρ:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\mathbf{v} = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla w + \frac{1}{4\pi\rho}[\operatorname{rot}\mathbf{H}\times\mathbf{H}], \tag{2}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{v}\times\mathbf{H}]. \tag{3}$$

Here, $\mathbf{v}$ is the plasma flow velocity, and w is the enthalpy related to the pressure $p = p(\rho)$ and the internal energy ε by the relationships

$$dw = \frac{dp}{\rho}, \quad w = \frac{\partial}{\partial \rho}\varepsilon(\rho).$$

A variational principle for this system of equations can be formulated as follows. First, it can be seen from Eqs. (1)–(3) that the vector $\mathbf{H}/\rho$ moves with the "liquid current" line; in other words, each field line moves with the particles occurring on this line, which corresponds to a well known concept of the "frozen-in" magnetic field (see, e.g., [16]). This circumstance allows the magnetic field $\mathbf{H}$ and the plasma density ρ] to be considered as generalized coordinates.

To formulate the variational principle, we will use a known expression for the Lagrangian of the electromagnetic field containing particles of a liquid [17]. We will write an expression for the Lagrangian L with neglect of a contribution due to the electric field relative to that due to the magnetic field, since $E \sim (V/c)H \ll H$. Taking into account relationships expressed by Eqs. (1) and (3) and the fact that $\operatorname{div}\mathbf{H} = 0$, we can write

$$L = \frac{\rho \mathbf{v}^2}{2} - \varepsilon(\rho) - \frac{H^2}{8\pi} + \mathbf{S}\cdot\left(\frac{\partial \mathbf{H}}{\partial t} - \operatorname{rot}[\mathbf{v}\times\mathbf{H}]\right) + \Phi\left(\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\mathbf{v}\right) + \psi \operatorname{div}\mathbf{H}.$$

where $\mathbf{S}$, Φ, and ψ are the Lagrange multipliers. Now we can use the so determined Lagrangian to introduce the functional of action

$$I = \int L dt d\mathbf{r},$$

the variation of which with respect to variables $\mathbf{v}$, ρ, and $\mathbf{H}$ leads to the following set of equations:

$$\rho\mathbf{v} = \mathbf{H}\times\operatorname{rot}\mathbf{S} + \rho\nabla\Phi, \tag{4}$$

$$\frac{\partial \Phi}{\partial t} + (\mathbf{v}\cdot\nabla)\Phi - \frac{v^2}{2} + w(\rho) = 0, \tag{5}$$

$$\frac{\partial \mathbf{S}}{\partial t} + \frac{\mathbf{H}}{4\pi} - \mathbf{v}\times\operatorname{rot}\mathbf{S} + \nabla\psi = 0. \tag{6}$$

The first equations suggests the change of variables, whereby the velocity $\mathbf{v}$ is expressed in terms of the new variables $\mathbf{S}$ and Φ. It must be emphasized that this change is not single-valued, since we may a vector $\mathbf{S}_0$ to $\mathbf{S}$ and a scalar Φ_0 to Φ such that

$$\mathbf{H}\times\operatorname{rot}\mathbf{S}_0 + \rho\nabla\Phi_0 = 0.$$

The two other equations, (5) and (6), represent the Bernoulli equation for the potential Φ and the equation of motion for the new vector $\mathbf{S}$ with an unknown potential ψ. The latter potential is set by specifying the calibration of vector $\mathbf{S}$. For example, the Coulomb calibration ($\operatorname{div}\mathbf{S} = 0$) determined ψ to within an arbitrary solution ψ_0 of the Laplace equation $\Delta\psi_0 = 0$:

$$\psi = \frac{1}{\nabla}\operatorname{div}[\mathbf{v}\times\operatorname{rot}\mathbf{S}] + \psi_0.$$

In particular, if $\mathbf{v} \longrightarrow 0$, $\mathbf{H} \longrightarrow \mathbf{H}_0$, and $\rho \longrightarrow \rho_0$ for $r \longrightarrow \infty$, the term ψ_0 is conveniently selected so that $S \longrightarrow 0$ for $r \longrightarrow \infty$. Then

$$\psi_0 = -\frac{\mathbf{H}_0\cdot\mathbf{r}}{4\pi}.$$

Now we have to check that system (4)–(6) does not contradict to the set of MHD equations. Substituting (4) into the equation of motion (2) and making simple transformations, we obtain

$$\nabla\left(\frac{\partial \Phi}{\partial t} + (\mathbf{v}\cdot V)\Phi - \frac{\mathbf{v}^2}{2} + w(\rho)\right) + \left[\frac{\mathbf{H}}{\rho}\times\operatorname{rot}\left\{\frac{\partial S}{\partial t} + \frac{\mathbf{H}}{4\pi} - \mathbf{v}\times\operatorname{rot}\mathbf{S}\right\}\right] = 0.$$

By virtue of Eqs. (5) and (6), this equation turns into identity. Thus, we have proved that the new system of Eqs. (1), (3), (5), and (6) is equivalent to the set of MHD equations. Indeed, any solution of this system generates, by virtue of (4), a solution to the MHD equations. If we assume uniqueness of the Cauchy problem for systems (1)–(4) and (1), (3), (5) and (6), the inverse statement is also valid: for any solution to Eqs. (1)–(4) we can find a certain class of solutions to system (1), (3), (5), and (6). Indeed, this is achieved by constructing all possible sets of $\mathbf{S}$ and Φ satisfying Eq. (4) for a given set of $\mathbf{v}$, $\mathbf{H}$, ρ at a given time instant t_0 and taking these $\mathbf{S}$ and Φ values as the initial conditions for system (1), (3), (5), and (6).

Once the Lagrange function is known, we can define the generalized momenta and construct a system Hamiltonian:

$$\mathcal{H} = \int(\mathbf{S}\cdot\mathbf{H}_t + \Phi\rho_t - L)d\mathbf{r} = \int\left\{\frac{\rho\mathbf{v}^2}{2} + \varepsilon(\rho) + \frac{\mathbf{H}^2}{8\pi} - \psi\operatorname{div}\mathbf{H}\right\}d\mathbf{r},$$

which coincides in magnitude with the total energy. The equations of motion (1), (3), (5), and (6) represent the Hamilton equations

$$\frac{\partial \rho}{\partial t} = \frac{\delta\mathcal{H}}{\delta\Phi}, \quad \frac{\partial \Phi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\rho}, \quad \frac{\partial \mathbf{H}}{\partial t} = \frac{\delta\mathcal{H}}{\delta\mathbf{S}}, \quad \frac{\partial \mathbf{S}}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\mathbf{H}}, \tag{7}$$

where the variables (ρ, Φ and ($\mathbf{H}$, $\mathbf{S}$) represent the pairs of canonically conjugated values.

The change of variables defined by Eq. (4) and the canonical description (7) were originally introduced for the magnetic hydrodynamics in [13]. The transformation (4) is an analog for the Clebsch representation in the ideal hydrodynamics; accordingly, the variables $\mathbf{H}$ and $\mathbf{S}$ in Eq. (4) play the role of the Clebsch variables (for the latter, see [17, 18] and a recent review [14]). Later, the same substitution was employed by Frenkel *et al.* in [18], where the velocity vector and the magnetic field were expressed through the scalar Clebsch variables; using simple transformations, this reduces to Eq. (4).

The MHD flows described by Eq (4), as well as the flows in the ideal liquid parametrized by the Clebsch variables, represent a partial flow type. For such MHD flows, the topological invariant of the magnetic field line and vorticity entanglement

$$I = \int(\mathbf{v}\cdot\mathbf{H})d\mathbf{r}$$

is identically equal to zero.

Vladimirov and Moffat [20] suggested an analog of the Weber transform for the ideal MHD flows:

$$\mathbf{v} = u_{0k}(\mathbf{a})\nabla a_k + \nabla\Phi + \frac{1}{\rho}\mathbf{H}\times\mathrm{rot}\mathbf{S}. \qquad (8)$$

where $\mathbf{a} = \mathbf{a}(\mathbf{r}, t)$ are the Lagrange markers of liquid particles (this is an inverse transformation with respect to $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$ determining the trajectory of a particle with the marker $\mathbf{a}$) and $\mathbf{u}_0(\mathbf{a})$ is a new Lagrange invariant.

The Weber transform (8) is a transformation of the general type containing the substitution (4) in a particular case of $\mathbf{u}_0 = 0$, which was not taken into account in [20]. The equations of motion for the potentials Φ and $\mathbf{S}$ have the same form as Eqs. (5) and (6). If $\Phi = 0$ and $\mathbf{S} = 0$ at $t = 0$, then $\mathbf{u}_0(\mathbf{a})$ is the initial velocity. It should be noted that it is the first term in (8) that ensures a nonzero value of the topological invariant I (this term is nonlinear if (8) is expanded in powers of small amplitude). Recently, Ruban [21] (see also [22]) elucidated a physical meaning of the new vector field $\mathbf{S}$. According to this, the quantity @@@curl$\mathbf{S}$ can be expressed through the displacement d between electron and ion (considered as liquid particles) at a point r at the time instant t (the initial coordinates are assumed to coincide):

$$@@@\mathrm{rot}\mathbf{S} = \frac{e}{Mc}\mathbf{d}\frac{\rho(\mathbf{r}, t)}{\rho_0(\mathbf{a})}.$$

Here M and e are the ion mass and charge, respectively, and $\rho_0(\mathbf{a})$ is the initial distribution of the plasma density.

Introduction of the canonical variables allows us to classify and study all nonlinear processes in a conventional manner, using the perturbation theory with respect to small wave amplitudes. For this purpose, it is necessary to expand the velocity and internal energy in Eq. (8) in powers of the canonical variables. If the plasma is placed into a homogeneous external magnetic field $\mathbf{H}_0$, the approximation linear in the wave amplitude must retain the terms linear in Φ and $\mathbf{S}$ and ignore the first (nonlinear) term in (8). As a result, the velocity expansion can be written as

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \dots \qquad (9)$$

where the first-order term is

$$\mathbf{v}_0 = \frac{1}{\rho_0}\mathbf{H}_0\times\mathrm{rot}\mathbf{S} + \nabla\Phi.$$

Three independent pairs ($\mathrm{div}\mathbf{H} = \mathrm{div}\mathbf{S} = 0$) of the canonically conjugated quantities correspond to the waves of three types. In the linear approximation, these waves do not interact with each other. The dispersion and polarization laws can be determined from an analysis of the quadratic (in powers of the canonical variables) Hamiltonian $\mathcal{H}_0$. The three-wave interaction corresponds to a cubic term, the magnitude of which is determined by a quadratic (in the wave amplitude) correction to the velocity:

$$\mathbf{v}_1 = \frac{\rho_1}{\rho_2}\mathbf{H}_0\times\mathrm{rot}\mathbf{S} + \frac{1}{\rho_0}\mathbf{h}\times\mathrm{rot}\mathbf{S}.$$

which takes into account only the "wave" degrees of freedom and neglects the first term in (8). Here, $\mathbf{h}$ and ρ_1 are the deviations of the magnetic field strength and the plasma density from the corresponding equilibrium values $\mathbf{H}_0$ and ρ_0. As a result, the Hamiltonian of the medium can be also written as an expansion in powers of the wave amplitude

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_3 + \dots, \qquad (10)$$

with the quadratic Hamiltonian

$$\mathcal{H}_0 = \int\left\{\frac{\rho_0\mathbf{v}_0^2}{2} + \frac{\mathbf{h}^2}{8\pi} + c_s^2\frac{\rho_1^2}{2\rho_0}\right\}d\mathbf{r},$$

and the cubic Hamiltonian

$$\mathcal{H}_3 = \int\left\{\rho_0(\mathbf{v}_0\cdot\mathbf{v}_1) + \frac{\rho_1}{2}v_0^2 + qc_s^2\frac{\rho_1^3}{2\rho_0^2}\right\}dr.$$

In these expressions, the squared sound velocity c_s^2 and the dimensionless coefficient q appeared as a result of expansion of the internal energy ε in powers of ρ_1:

$$\Delta\varepsilon(\rho) = \frac{\rho_0 c_s^2}{2}\left\{\left(\frac{\rho_1}{\rho_0}\right)^2 + q\left(\frac{\rho_1}{\rho_0}\right)^3 + \dots\right\}.$$

Now let us accomplish the Fourier transform with respect to coordinates and pass to the new variables $a_j(k)$ ($j = 1, 2, 3$), which yields

$$\mathbf{h}(k) = \mathbf{e}_1(k)\sqrt{2\pi\omega_1}(a_1(k) + a_1^*(-k))$$

$$+ \mathbf{e}_2(k)\sum_{l=2,3}\lambda_l\sqrt{2\pi\omega_l}(a_l(k) + a_1^*(-k)),$$

$$\mathbf{S}(k) = -i\mathbf{e}_1(k)\frac{1}{\sqrt{8\pi\omega_1}}(a_1(k) - a_1^*(-k))$$
$$- i\mathbf{e}_2(k)\sum_{l=2,3}\lambda_l\frac{1}{\sqrt{8\pi\omega_l}}(a_l(k) - a_l^*(-k)), \qquad (11)$$
$$\rho_1(k) = \sum_{l=2,3}\left(\frac{\rho_0\omega_l}{2c_s^2}\right)^{1/2}\mu_l(a_k(l) + a_{-k}^*(l)),$$
$$\Phi k = -i\sum_{l=2,3}\left(\frac{c_s^2}{2\rho_0\omega_l}\right)^{1/2}\mu_l(a_l k - a_l^*(-k)).$$

Here

$$\omega_1(k) = |\mathbf{k}\cdot\mathbf{V}_A|,$$
$$\omega_{2,3}(k) = \frac{1}{2}\Big|\sqrt{k^2V_A^2 + k^2c_s^2 + 2(\mathbf{k}\cdot\mathbf{V}_A)kc_s}$$
$$\pm\sqrt{k^2V_A^2 + k^2c_s^2 - 2(\mathbf{k}\cdot\mathbf{V}_A)kc_s}\Big|$$

are the dispersion laws for the Alfvén waves ($j = 1$) and the fast ($j = 2$) and slow ($j = 3$) magnetoacoustic waves; the corresponding unit polarization vectors are

$$\mathbf{e}_1(k) = \frac{[\mathbf{k}\times\mathbf{n}_0](\mathbf{k}\cdot\mathbf{n}_0)}{|\mathbf{k}\times\mathbf{n}_0||\mathbf{k}\cdot\mathbf{n}_0|}, \quad \mathbf{e}_2(k) = \frac{\mathbf{k}\times[\mathbf{k}\times\mathbf{n}_0]}{k|\mathbf{k}\times\mathbf{n}_0|},$$

($\mathbf{n}_0 = \mathbf{H}_0/H_0$ is the unit vector of the average magnetic field);

$$\mathbf{V}_A = \frac{\mathbf{H}_0}{4\pi\rho_0)^{1/2}}$$

is the Alfvén velocity; and

$$\lambda_2 = -\mu_3 = -\left(1 - \frac{\omega_2^2 - k^2c_s^2}{\omega_2^2 - k^2c_s^2}\right)^{1/2},$$
$$\lambda_3 = \mu_2 = \left(1 - \frac{\omega_2^2 - k^2c_s^2}{\omega_3^2 - k^2c_s^2}\right)^{-1/2}.$$

The change of variables $a_k(j)$ represents a canonical U–V transform diagonalizing the Hamiltonian $\mathcal{H}_0$:

$$\mathcal{H}_0 = \sum_j\int\omega_j(k)a_j(k)a_l^*(k)d\mathbf{k}.$$

The amplitudes $a_k(j)$ play the role of normal variables, for which the equations of motion acquire the canonical form

$$\frac{\partial a_j(k)}{\partial t} = -i\frac{\delta\mathcal{H}}{\delta a_j^*(k)}.$$

In the linear approximation, the quantities $a_k(j)$ obey the following equations:

$$\frac{\partial a_j(k)}{\partial t} + i\omega_j(k)a_j(k) = 0,$$

which imply that the amplitude modulus $|a_i(k)|$ does not change with the time t, while the phase grows linearly with t.

In order to find an expression for the interaction Hamiltonian in terms of the variables $a_j(k)$, it is necessary to substitute transform (11) into expansion (10). As a result, the Hamiltonian of the wave interaction has the form of an integro-power series with respect to $a_j(k)$. In the lowest order with respect to the wave amplitude, the principal nonlinear process is the three-wave resonance interaction corresponding to the Hamiltonian

$$\mathcal{H}_{\text{int}} = \frac{1}{2}\int\sum_{lmn}[V_{kk_1k_2}^{lmn}a_l^*(k)a_m(k_1)a_n(k_2) + \text{c.c.}]$$
$$\times\delta_{k-k_1-k_2}dkdk_1dk_2. \qquad (12)$$

This Hamiltonian is obtained by substituting transform (11) into the cubic Hamiltonian $\mathcal{H}_3$ and separating the corresponding resonance terms. The remaining terms in $\mathcal{H}_3$ are small and can be excluded with the aid of a canonical transformation (for more detail, see [14]). Note that calculation of the matrix elements $V_{kk_1k_2}^{ije}$ in this scheme is a purely algebraic procedure involving Fourier transform in the integrals, substitution of (11), and the symmetrization with respect to variables $a_k(i)$ [for example, with respect to (k_1, m) and (k_2, n) in Eq. (12).

3. AVERAGED EQUATIONS

Expressions for the dispersion laws and the matrix elements of interaction can be significantly simplified in the case of a plasma with small $\beta = 8\pi nT/H^2$ (representing the ratio of the thermal plasma pressure nT to the magnetic field pressure $H^2/8\pi$). The condition $\beta \ll 1$ implies that $V_A \gg c_s$ In this limit, the fast magnetoacoustic waves possess an isotropic dispersion law $\omega_2 = kV_A$ and their phase (and group) velocity coincides with the group velocity of the Alfvén waves. In this linear approximation, the velocity of plasma in the Alfvén waves and the fast magnetoacoustic waves is determined by the following formula:

$$\mathbf{v}_{HF} = \frac{1}{\rho_0}\mathbf{H}_0\times\text{rot}\mathbf{S}.$$

The potential part of the plasma velocity $\nabla\Phi$ is a small quantity with respect to the parameter β. In contrast, the main contribution to the velocity of slow magnetoacoustic waves is due to the potential part. For this reason, this velocity is directed along the magnetic field $\mathbf{H}_0$,

$$\mathbf{v}_s = \mathbf{n}_0 \frac{\partial \Phi}{\partial z}, \tag{13}$$

and the dispersion of the slow magnetoacoustic waves becomes strongly anisotropic:

$$\omega_3 \equiv \Omega_s = |k_s| c_s. \tag{14}$$

The transverse velocity components in these waves $[\mathbf{H}_0 \times \mathrm{rot}\mathbf{S}]/\rho_0$ are compensated by the term $\nabla_\perp \Phi$.

If the plasma is collisionless and strongly nonisothermal ($T_e \gg T_i$), the slow magnetoacoustic waves represent the magnetized ion-sound waves (considered in more detail in [7]). In this case, the sound velocity in Eq. (14) can be expressed as

$$c_s = \sqrt{T_e/M}.$$

As for a nonlinear interaction of the MHD waves, the principal nonlinear mechanism is the resonance scattering of fast magnetoacoustic and Alfvén waves on the slow magnetoacoustic waves. This is clearly seen from a comparison of the calculated matrix elements V^{lmn} in the Hamiltonian (12). In this process, the former waves (called the A-waves) appear as high-frequency (HF) with respect to the latter (simply referred to below as the acoustic waves, or S-waves). This conclusion follows from an analysis of the resonance conditions for this decay process:

$$\omega_A(k) = \omega_A(k_1) + \Omega_s(k_2), \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2. \tag{15}$$

It is quite easy to see qualitatively how this interaction proceeds in the system studied. As an A-wave packet propagates in the plasma, the average characteristics (plasma density and velocity) slowly vary under the action of these waves. Owing to this, the average Alfvén velocity differs from a local value by the quantity

$$\Delta V_A = -V_A \rho_{1s}/2\rho_0,$$

where ρ_{1s} is the low-frequency (LF) density variation. As a result, frequencies of the A-wave acquire an increment $\Delta\omega_\rho \sim k\Delta V_A$. Due to a slow motion with the drift velocity v_D, the A-wave frequency changes by $\Delta\omega_D \sim kv_D$ (Doppler effect). The ratio of the two frequencies ($\Delta\omega_D$ and $\Delta\omega_\rho$) is a small quantity with respect to the parameter c_s/V_A. Therefore, a principal interaction is the scattering on the LF density fluctuations. Note that the LF characteristics of the plasma change under the HF force action of the A-waves.

An expression for the HF force is most simply derived by averaging the Hamiltonian over the HF oscillations. Upon this averaging, the Hamiltonian acquires the following form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{int}}, \tag{16}$$

where

$$\mathcal{H}_0 = \int \left\{ \frac{1}{2\rho_0} \langle [\mathbf{H}_0 \times \mathrm{rot}\mathbf{S}]^2 \rangle + \frac{\langle \mathbf{h}^2 \rangle}{8\pi} \right\} d\mathbf{r}$$

$$+ \int \left\{ \frac{\rho_0 \Phi_z^2}{2} + c_s \frac{\rho_{1s}^2}{2\rho_0} \right\} d\mathbf{r},$$

$$\mathcal{H}_{\mathrm{int}} = -\int \frac{\rho_{1s}}{2\rho_0^2} \langle [\mathbf{H}_0 \times \mathrm{rot}\mathbf{S}]^2 \rangle d\mathbf{r}.$$

(the angle brackets denote the averaging over high frequencies). The first integral in $\mathcal{H}_0$corresponds to the A-waves, while the second represents the acoustic oscillations of the magnetized plasma. Variation of the interaction Hamiltonian with respect to ρ_{1s} yields the following expression for the HF interaction potential:

$$U \equiv M \frac{\delta \mathcal{H}_{\mathrm{int}}}{\delta \rho_{1s}} = -\frac{1}{2Mn_0^2} \langle [\mathbf{H}_0 \times \mathrm{rot}\mathbf{S}]^2 \rangle. \tag{17}$$

According to this, the equation of motion for the potential Φ_s is as follows:

$$\frac{\partial \Phi_s}{\partial t} + c_S^2 \frac{\rho_{1s}}{\rho_0} = \frac{\langle [\mathbf{H}_0 \times \mathrm{rot}\mathbf{S}]^2 \rangle}{2\rho_0^2}. \tag{18}$$

It is important to note that the HF potential (17) is negative, which implies that the HF force acting in the region of localization of the A-waves leads to the appearance of the density "humps" (instead of the "dips" observed for the interaction of ion-sound waves with the Langmuir waves [23]).

The equations of motion for the slow wave component are closed by the equation of continuity for ρ_{1s} According to (13), this equation can be written as

$$\frac{\partial \rho_{1s}}{\partial t} + \rho_0 \frac{\partial^2 \Phi_s}{\partial z^2} = 0. \tag{19}$$

Using Eq. (18) and (19), we obtain

$$\frac{\partial^2 \rho_{1s}}{\partial t^2} - c_s \frac{\partial^2 \rho_{1s}}{\partial z^2} = -\frac{1}{2\rho_0} \frac{\partial^2}{\partial z^2} \langle [\mathbf{H}_0 \times \mathrm{rot}\mathbf{S}]^2 \rangle. \tag{20}$$

In order to derive equations for the A-waves, it is necessary to perform averaging of the interaction Hamiltonian $\mathcal{H}_{\mathrm{int}}$. This is achieved by retaining terms of the type $a_\lambda^* a_\lambda$ ($\lambda = 1, 2$ is the HF wave number):

$$H_{int} = -\int \frac{\rho_{1s}(k)}{2\rho_0} \sum_{\lambda\lambda_1} F_{kk_1}^{\lambda\lambda_1} a_\lambda^*(k) a_{\lambda_1}(k_1) \delta_{k-k_1-\kappa} d\mathbf{k} d\kappa.$$

where

$$F_{kk_1}^{\lambda\lambda_1} = (\omega_\lambda(k)\omega_{\lambda_1}(k_1))^{1/2} \mathbf{n}_\lambda(k) \cdot \mathbf{n}_{\lambda_1}(k_1),$$

$$\mathbf{n}_2 = \frac{\mathbf{k}_\perp}{k_\perp}, \quad \mathbf{n}_1 = -\mathbf{n}_2 \times \mathbf{n}_0.$$

As a result, the A-wave equations acquire the following form:

$$\frac{\partial a_\lambda(k)}{\partial t} + i\omega_\lambda(k)a_\lambda(k) = -i\frac{\delta H_{\text{int}}}{\delta a_\lambda^*(k)}, \quad \lambda = 1, 2. \quad (21)$$

A collisionless isothermal ($T_e \approx T_i$) plasma features no slow magnetoacoustic oscillations as a result of the strong Landau damping on the ions. Accordingly, the A-wave decay interaction [15] changes to the induced scattering of A-waves on the ions. In this case, Eqs. (20) have to be replaced by a system of the kinetic drift equation [24] for a slow variation of the ion distribution function f_i (cf. [25]),

$$\frac{\partial f_i}{\partial t} + v_z\frac{\partial f_i}{\partial z} - \frac{1}{M}\frac{\partial}{\partial z}(e\tilde{\varphi} + U)\frac{\partial f_0}{\partial v_z} = 0 \qquad (22)$$

and the condition of quasineutrality for the slow motions ($\Omega_k = k_z c_S \ll \ddot{\omega}_{pi}$),

$$\delta n_i = \int f_i d\mathbf{v} = \frac{n_0}{T_e}e\tilde{\varphi} = \frac{\rho_{1s}}{M}, \qquad (23)$$

where f_0 is the equilibrium ion distribution function and $\tilde{\varphi}$ is the LF electrostatic potential. The equations of motion for the A-waves retain the form of Eq. (21) and the plasma density is linearly expressed via the HF potential in terms of Green's function for the system of Eqs. (22) and (23):

$$G_{\kappa\Omega} \equiv \frac{\rho_{1s}(\kappa, \Omega)}{U_{\kappa\Omega}} = -\frac{n_0\kappa^2}{\omega_{pi}^2}\frac{\epsilon_e\epsilon_i}{\epsilon_e + \epsilon_i}. \qquad (24)$$

Here, $\rho_{1s}(\kappa, \Omega)$ and $U_{\kappa\Omega}$ are the Fourier images of the LF density and HF potential, respectively, and $\epsilon_{e,\,i}$ are the partial permittivities of electrons and ions:

$$\epsilon_e = \frac{1}{\kappa^2 r_d^2},$$

$$\epsilon_i = \frac{4\pi e^2}{M\kappa^2}\int\frac{\kappa_z(\partial f_0/\partial v_z)}{\Omega - k_z v_z}d\mathbf{v},$$

($r_d^2 = T_e/4\pi n_0 e^2$ is the squared Debye radius). In a strongly nonisothermal plasma ($T_e \gg T_i$), Green's function (24) converts into

$$G_{\kappa\Omega} = \frac{n_0\kappa_z^2}{\Omega^2 - \kappa_z^2 c_s^2},$$

which coincides with the expression for the function determined by Eq. (20).

The system of equations (22)–(24) completely describes the interaction of A-waves in a strongly magnetized plasma with an arbitrary ratio of the electron and ion temperatures. However, because of the Landau damping on ions, the Hamiltonian $H_0 + H_{\text{int}}$ is no longer conserved.

4. INSTABILITY OF A MONOCHROMATIC WAVE

Now we proceed to analysis of the equations derived in the preceding section. First, let us consider the behavior of a narrow A-wave packet. A qualitative pattern of this process can be outlined by studying the stability of a monochromatic A-wave. For simplicity, we will restrict the consideration to the stability of an Alfvén wave in a hydrodynamic limit. For a collisionless plasma, this implies that a phase velocity of Ω/k_z of beats in the A-wave exceeds the thermal ion velocity v_{Ti}. Under these conditions, we may neglect the Landau damping on ions for the slow acoustic oscillations and use Eqs. (20) or (24). It should be born in mind that the acoustic waves in a strongly nonisothermal plasma represent intrinsic oscillations, whereas the sound generated in an isothermal plasma ($T_e \approx T_i$) represents induced oscillations in the plasma density. However, under the condition $\Omega/k_z \gg v_{Ti}$, the hydrodynamic description is applicable in both cases.

It is convenient to express ρ_{1s} through the normal variables $a_3(k) \equiv b_k$:

$$\rho_{1s}(k) = \left(\frac{\rho_0\Omega_k}{2c_s}\right)^{1/2}(b_k + b_{-k}^*).$$

Equations for the new variables $b(k)$ are obtained by variation of the total Hamiltonian $H_0 + H_{\text{int}}$:

$$\frac{\partial b_k}{\partial t} + i\Omega(k)b_k = -i\frac{\delta H_{\text{int}}}{\delta b_k^*}. \qquad (25)$$

A monochromatic Alfvén wave corresponds to the following equation of Eqs. (21) and (25):

$$a_\lambda(k) = \frac{A}{\omega_0^{1/2}}\delta_{\lambda 1}\exp(-i\omega_0 t)\delta_{k-k_0}, \quad b_k = 0,$$

$$\omega_0 = \omega_1(k_0).$$

Here, the Alfvén wave amplitude is selected so that the value $W = |A|^2$ would coincide with the energy density of the oscillations.

Upon linearizing Eqs. (22)–(24) relative to the exact solution and taking perturbations in the form of

$$\delta a_\lambda(k) \propto \exp(-i(\Omega + \omega_0)t)\delta_{k-k_0-\kappa},$$

$$\delta a_\lambda^*(k) \propto \exp(-i(\Omega - \omega_0)t)\delta_{k-k_0+\kappa},$$

we obtain the following dispersion relationship for Ω:

$$\frac{WG}{4Mn_0^2\omega_0}\sum_\lambda\left\{\frac{\left|F^{1\lambda}_{k_0,\,k_0+\kappa}\right|^2}{\Omega + \omega_0 - \omega_\lambda(k_0+\kappa)} + \frac{\left|F^{1\lambda}_{k_0,\,k_0+\kappa}\right|^2}{-\Omega + \omega_0 - \omega_\lambda(k_0-\kappa)}\right\} = 1. \qquad (26)$$

Now we will present the results of investigation of the dispersion relationship (26) in various special cases depending on the oscillation energy density W and the ratio of electron and ion temperatures in the plasma.

For $T_e \gg T_i$ and sufficiently small oscillation amplitudes, the A-wave exhibits decay instability with the ion-sound excitation [10]. For this instability, the frequency Ω can be expressed through the matrix element of the decay interaction

$$V_{kk_1k_2}^{\lambda\lambda_1} = \left(\frac{\Omega(k_2)}{8\rho_0 c_s^2}\right)^{1/2} F_{kk_1}^{\lambda\lambda_1} \tag{27}$$

and the energy density W as

$$\Omega = \frac{1}{2}[\omega_0 - \omega_\lambda(k_0-\kappa) + \Omega(\kappa)]$$
$$\pm\left\{\frac{1}{4}[\omega_0 - \omega_\lambda(k_0-\kappa) - \Omega(\kappa)]^2 - \frac{W}{\omega_0}\left|V_{k_0,\,k_0-\kappa,\,\kappa}^{1\lambda}\right|^2\right\}^{1/2}. \tag{28}$$

From this expression, it follows that the instability takes place in the vicinity of the resonance surface

$$\omega_0 = \omega_\lambda(k_0-\kappa) + \Omega(\kappa) \tag{29}$$

with a maximum increment

$$\Gamma = \left[\frac{W}{8nT}\frac{\Omega_\kappa}{\omega_0}\left|F_{k_0,\,k_0-\kappa}^{1\lambda}\right|^2\right]^{1/2}. \tag{30}$$

The increment width with respect to the frequency is on the order of the maximum increment value (30).

Since the matrix element is proportional to the square root of the slow sound frequency, the maximum increment on the resonance surface (29) is reached for the maximum value of $|\kappa_z|$. Upon decay into the Alfvén wave and the slow acoustic wave,

$$\max|\kappa_z| \approx 2|k_{0z}|,$$

which implies that the secondary Alfvén wave propagates in the direction opposite to that of the primary (exciting) Alfvén wave. The character of the decay instability is typical of the Mandel'shtam–Brillouin scattering, the matrix element for which is proportional to the square root of the momentum of light transmitted to the acoustic waves as a result of scattering. This accounts for the maximum backscattering of light.

Now we can readily investigate the decay instability in all other channels of the decay process $A \longrightarrow A + S$. In all these cases, the increments are on the same order of magnitude as the increment determined by formula (30):

$$\Gamma \sim (\omega_0\Omega_s W/nT)^{1/2}.$$

This instability takes place for $W/nT < \beta^{1/2}$. As the W/nT ratio increases, the decay instability is modified. For $W/nT > \beta^{1/2}$, we may neglect Ω_s^2 in comparison to Ω^2 in relationship (26). Then, the unstable wavevectors are lying on the surface

$$\omega_1(k_0) = \omega_\lambda(k_0-\kappa).$$

This instability is referred to as the modified decay instability [15, 26]. In the case of interaction between the Alfvén waves and the slow acoustic waves, this instability has an increment reaching maximum at $\kappa_z = 2k_{0z}$:

$$\Gamma \approx \frac{\sqrt{3}}{2}\omega_0\left(\frac{W}{\rho_0 V_A^2}\right)^{1/3}. \tag{31}$$

Since this value is independent of the temperature, the same instability may take place for $W/nT > 1$ as well (up to $W/nT \sim 1/\beta$, when the main assumption of adiabatically, $\Gamma \sim \omega_0$, fails to be valid).

In the other channels, the behavior of instability with increasing parameter W/nT exhibits the same pattern: for $W/nT > \beta^{1/2}$, the increment reaches maximum at $\kappa \sim k_0$ and coincides in the order of magnitude with the value given by formula (31).

As can be readily seen, a decay instability with the increment (30) for an arbitrary channel $A \longrightarrow A + S$ belongs to instability of the convective type. According to relationship (28), the group velocities of the excited waves are significantly different from the group velocity of the primary (exciting) wave. Therefore, for the wave packet with a length L, the instability will be significant only provided L is sufficiently large so that the gain G would exceed the Coulomb logarithm Λ:

$$G = \Gamma L/V_A \approx \Lambda.$$

For smaller L values, the decay instability will be not manifested since the perturbation amplitude acquires only a small increment during the time required for the perturbation to travel through the entire packet length. In this case, the wave packet dynamics is determined by slower processes. Among these, the most important are related to the unstable perturbations propagating together with the wave packet. Should it be a decay instability, this instability must be absolute (in the coordinate system moving with the wave packet). This is one of the possible factors for a collapse of the fast magnetoacoustic waves producing a special effect on the structure of collisionless shock waves in a plasma [27, 28]. The collapse of fast magnetoacoustic waves arises a result of a three-weave interaction involving only the fast magnetoacoustic waves.

5. THE KOLMOGOROV SPECTRA

In the preceding section, we have considered the instability of a wave packet narrow in the k-space. Upon the decay of a monochromatic wave obeying the resonance conditions (29) (i.e., reaching the maximum increment given by formula (30)), the sum of phases of the excited waves ($\phi_A + \phi_s$) is strictly related to the pumping wave phase (ϕ_0):

$$\phi_0 + \pi/2 = \phi_A + \phi_s.$$

The phase difference in the pair of excited waves with a fixed wavevector κ is arbitrary. As can be readily checked, the above phase correlation is lost on deviating from the resonance conditions (29). Both these factors introduce an element of stochasticity into the system of interacting triads related to the pumping wave. Thus, each triad is characterized by a single random phase. In the next stage (secondary cascade), new random phases are added and the memory of a coherent pumping wave is lost. Upon numerous repeats of this process, the system must pass to a turbulent state in which the wave phases can be considered random. Therefore, the randomization time must be equal to several times the reciprocal increment give by formula (30). This scenario of transition to a turbulent state seems to be quite realistic. A series of numerical experiments were aimed at the verification of this hypothesis (see, e.g., [29, 30].

Based on the above considerations, it is clear that a regime of developed turbulence can be expected to possess a wide spectrum of waves. This spectrum can be statistically described in terms of correlation functions. For the waves of small intensity, is sufficient to restrict the consideration to pair correlation functions obeying the kinetic equations. This regime is referred to as weakly turbulent.

In the case of a weak MHD turbulence ($\beta \ll 1$), we obtain three pair correlation functions determined by the formulas

$$\langle a_\lambda(k)a^*_{\lambda_1}(k_1)\rangle = N_k^\lambda \delta_{\lambda\lambda_1}\delta_{k-k_1}, \quad \langle b_k b^*_{k_1}\rangle = n_k\delta_{k-k_1},$$

Here, the coefficients N_k^λ and n_k having the sense of occupation numbers obey the following system of equations:

$$\dot{n}_k = 2\pi\int |V_{k_1k_2k}|^2(N_{k_1}N_{k_2} - n_kN_{k_1} + n_kN_{k_2}) \times \delta_{k+k_1-k_2}\delta_{\Omega+\omega_1-\omega_2}dk_1dk_2, \tag{32}$$

$$\begin{aligned}\dot{N}_k = {} & 2\pi\int |V_{kk_1k_2}|^2(N_{k_1}n_{k_2} - N_kn_{k_2} - N_kN_{k_1}) \\ & \times\delta_{k-k_1-k_2}\delta_{\omega-\omega_1-\Omega_2}dk_1dk_2 \\ & - 2\pi\int |V_{k_1kk_2}|^2(N_kn_{k_2} - N_{k_1}n_{k_2} - N_kN_{k_1}) \\ & \times\delta_{k_1-k-k_2}\delta_{\omega_1-\omega-\Omega_2}dk_1dk_2.\end{aligned} \tag{33}$$

where $\omega \equiv \omega(k)$, $\omega_1 \equiv \omega(k_1)$, etc. In these equations (and below) we omitted the summation with respect to λ, which can be restored by substituting

$$dk_1 \longrightarrow \sum_n dk_1, \quad N_k \longrightarrow N_k^\lambda,$$

$$\omega_k \longrightarrow \omega_{k\lambda}, \quad V_{kk_1k_2} \longrightarrow V_{kk_1k_2}^{\lambda\lambda_1}$$

etc.

Equations (32) and (33) assume that the nonlinear wave interaction is weak. In this particular case, the most significant condition is

$$\Omega_s \gg 1/\tau,$$

where τ is the characteristic nonlinear time determined from the kinetic equations (32) and (33). In order to estimate τ, it is necessary to take into account that, in every wave decay event and the reverse (wave merging) process, the A-wave frequencies change by a small increment $\Delta\omega_A = \Omega_s \ll \omega_A$ so that the A-wave energy is redistributed over the spectrum in a diffusion manner. Taking his into account, we obtain the following estimate for τ:

$$\frac{1}{\tau} \sim \omega_A\frac{W}{\rho V_A^2}.$$

Note that this τ value is significantly greater than the characteristic time of randomization determined as the reciprocal increment Γ^{-1} determined by formula (30). Finally, the criterion can be written in the following form:

$$\frac{W}{\rho V_A^2} \ll \beta^{1/2}.$$

Now let us include the sources of turbulence and damping into Eqs. (32) and (33). For this purpose, we introduce the terms $\Gamma_k n_k$ and $\gamma_{k\lambda}N_{k\lambda}$ into the left-hand parts of these equations, respectively. It will be assumed that the domains of pumping (Γ_k, $\gamma_{k\lambda} > 0$) and damping (Γ_k, $\gamma_{k\lambda} < 0$) are separated in the k-space by an intermediate region (interval of inertia) in which ht turbulence dynamics is determined only by the wave interaction. If we can neglect the pumping and damping effects in the interval of inertia (which has to be proved), the n_k and $N_{k\lambda}$ distributions would be independent of the particular form of γ_k and Γ_k.

It should be recalled that determination of the turbulence spectrum (describing the distribution of pulsations over scales) in the theory of hydrodynamic turbulence is based on the two Kolmogorov hypotheses [1]. According to the first hypothesis concerning the automodel character of the turbulence spectrum, the spectrum in the interval of inertia is determined by a single quantity P representing a constant energy flux in the spectrum. The second hypothesis stipulates that the interaction of pulsations with different k values has a local character.

Applying these hypotheses to our situation, the turbulence spectra can be determined proceeding from the dimensionality considerations. In this case, the kinetic equations (32) and (33) reflect conservation laws—for the energy and the number of HF waves. Each of these laws must correspond to a Kolmogorov spectrum of its own. Indeed, a constant flux of the number of HF waves (N_k^λ)

$$P_N = \frac{\partial}{\partial t}\sum_\lambda N_{k\lambda}dk,$$

corresponds to the spectrum

$$N_{k\lambda} \propto P_N^{1/2}k^{-4}, \quad n_k \propto P_N^{1/2}k^{-4}, \tag{34}$$

while the constant energy flux

$$P_\varepsilon = \frac{\partial}{\partial t}\int\left(\omega_k n_k + \sum_\lambda \omega_{k\lambda}N_{k\lambda}\right)dk$$

corresponds to

$$N_{k\lambda} \propto P_\varepsilon^{1/2}k^{-3/2}, \quad n_k \propto P_\varepsilon^{1/2}k^{-3/2}. \tag{35}$$

Based on the conservation of the total number of HF waves and the energy in the interval of inertia, one may readily infer that the flux of the particle number N is directed toward small k, whereas the energy flux P_ε is directed toward large k. The rough approximation of the turbulence spectra given by formulas (34) and (35) may only provide for a correct description of the effect of wavenumber and the fluxes, but ignore the diffusion character of the wave decay process. It should be also recalled that the above conclusions are based on the hypothesis of a local character of the wave interaction.

The spectra (34) and (35) fail to describe fine details of the distribution functions (such as the angular dependence). Therefore, these functions are determined to within an arbitrary function of the angles. The angular dependence can be described by solving the exact equations (32) and (33). The solutions can be obtained, in particular, for the interaction of Alfvén waves with acoustic waves ($N_2 \equiv 0$). For this purpose, Eqs. (32) and (33) can be conveniently rewritten as

$$\dot{n}_k = -\int U_{k_2|kk_1}T_{k_2|kk_1}dk_1dk_2\mathrm{LS} \tag{36}$$

$$\dot{N}_k = \int(U_{k|k_1k_2}T_{k|k_1k_2} - U_{k_1|kk_2}T_{k_1|kk_2})dk_1dk_2, \tag{37}$$

where

$$U_{k|k_1k_2} = 2\pi\left|V^{11}_{k'k'_1k_2}\right|^2\delta_{k-k_1-k_2}\delta_{\omega-\omega_1-\omega_2},$$

$$T_{k|k_1k_2} = N_{k_1}n_{k_2} - N_kn_{k_2} - N_kN_{k_1}.$$

As can be readily checked, Eqs. (36) and (37) possess thermodynamically equilibrium solutions

$$N_k = \frac{N}{\omega_k\mu}, \quad n_k = \frac{T}{\Omega_k},$$

representing the Rayleigh–Jeans distributions for which the collisional term is zero.

In order to determine the nonequilibrium distributions, note that the function U (having the sense of the decay probability) possesses the following properties: U is a bihomogeneous function of variables k_α and $k_\perp$, the degree of homogeneity being +1 for all k_z and –2 for $k_\perp$. In addition, U is invariant with respect to rotation relative to the z axis coinciding with the direction of the magnetic field $\mathbf{H}_0$.

For these reasons, it is natural to seek for solutions in the form of

$$n_k = Ak_z^\alpha k_\perp^\beta, \quad N_k = Bk_z^\alpha k_\perp^\beta. \tag{38}$$

Consider stationary solutions to Eq. (37):

$$\int(U_{k|k_1k_2}T_{k|k_1k_2} - U_{k_1|kk_2}T_{k_1|kk_2}) = 0. \tag{39}$$

Let us map the integration domain (determined by the conservation laws) of the second integral (39) onto that of the first integral. This is conveniently performed by introducing the complex quantity $\zeta = k_x + ik_y$. Using this value, the conservation laws determining the integration domain of the second integral can be written as:

$$k_{z1} - k_z - k_{z2} = 0,$$

$$\zeta_1 - \zeta - \zeta_2 = 0,$$

$$\omega_1 - \omega - \Omega_2 = 0.$$

This domain is mapped onto that of the first integral (39) by the following conformal mapping with respect to variables k_z and ζ

$$= k_z\mathrm{DBE}\frac{k_z}{k'_z}, \quad \zeta = \zeta'\frac{\zeta}{\zeta}$$

$$k_{z1} = k_z\frac{k_z}{k'_z}, \quad \zeta_1 = \zeta\frac{\zeta}{\zeta'}, \tag{40}$$

$$k_{z2} = k''_z\frac{k_z}{k'_z}, \quad \zeta_2 = \zeta''\frac{\zeta}{\zeta'},$$

Here, each separate transformation is inversion: relative to the point k_z for the z-components of the wave vector and relative to the circumference of radius $|k_\perp|$ for the transverse components. As a result, the vector $\mathbf{k}$ transforms into $\mathbf{k}_1$, $\mathbf{k}_1$ into $\mathbf{k}$, and $\mathbf{k}_2$ into $\mathbf{k}_2$. Simultaneously, the z-components are stretched by the factor $|k_z/k_{z1}|$ and the transverse components, by the factor $|k_\perp/k_{1\perp}|$; this is complemented by rotation about the z axis through the angle $\arg(\zeta/\zeta_1)$.

As a result (with an allowance for the properties of U and T values), the integrand in the second integral (39) converts into the integrand of the first integrand multiplied by a power factor:

$$\int U_{k|k_1k_2}T_{k|k_1k_2}\left[1 - \left(\frac{k_z}{k_{z1}}\right)^{2\alpha+4}\left(\frac{k_\perp}{k_{\perp1}}\right)^{2\beta+4}\right]dk_1dk_2 = 0.$$

From this we may infer that, besides thermodynamically equilibrium solutions (for which T turns zero), there exist nonequilibrium solutions

$$n_k = Ak_z^{-2}k_\perp^{-2}, \quad N_k = Bk_z^{-2}k_\perp^{-2}, \tag{41}$$

which correspond to the solutions obtained previously proceeding from the dimensionality considerations for the constant particle number flux P_N. A relationship between the coefficients A and B in (41) is determined from the stationary equation (32) ($\partial/\partial t = 0$). From this we readily obtain an estimate $c_sA \sim V_AB$, which implies that the energies of the acoustic and Alfvén waves in the stationary case are on the same order of magnitude.

Note that the set of all mappings of the type (40) forms a group G, which is a direct product

$$G = G(1) \times G(2).$$

of groups $G(1)$ and $G(2)$ acting in one- and two-dimensional spaces, respectively. For power-type spectra, mappings of this type lead to factorization of the collisional term. The one-dimensional transformations (in the frequency space) for isotropic spectra were determined by Zakharov [5, 6, 23]. Generalizations of these transformations to the two- and tree-dimensional cases were formulated by Kats and Kontorovich [31]. Mappings (40) represent a partial case of the quasiconformal mapping.

In order to determine the second nonequilibrium solution (35), we introduce a quantity

$$\varepsilon_k = \omega_k N_k + \Omega_k n_k$$

representing the energy density in the k-space. According to (36) and (37), ε_k obeys the equation

$$\begin{aligned}\frac{\partial \varepsilon_k}{\partial t} = \int\{\omega_k U_{k|k_1k_2}T_{k|k_1k_2} - \omega_k U_{k_1|kk_2}T_{k_1|kk_2} \\ - \Omega_k U_{k_2|k_1k}T_{k_2|k_1k}\}dk_1dk_2.\end{aligned} \tag{42}$$

Let us find the stationary solutions to this equation in the same form (38) as above. To this end, we also map the integration domain of the second and third integrals (42) onto that of the first integral. For the second integral, the mapping coincides with (40); for the third integral, the required mapping is

$$\begin{aligned} k_z &= k_z''\frac{k_z}{k_z''}, \quad \zeta = \zeta''\frac{\zeta}{\zeta''}, \\ k_{z_1} &= k_z'\frac{k_z}{k_z''}, \quad \zeta_1 = \zeta'\frac{\zeta}{\zeta''}, \\ k_{z_2} &= k_z\frac{k_z}{k_z''}, \quad \zeta_2 = \zeta\frac{\zeta}{\zeta''}. \end{aligned} \tag{43}$$

Substituting (40) and (43) into the stationary equation (42), we obtain

$$\begin{aligned} 0 = \int |V_{kk_1k_2}|^2 \delta_{k-k_1-k_2}\delta_{\omega-\omega_1-\Omega_2}T_{k|k_1k_2}dk_1dk_2 \\ \times\left\{\omega(k) - \omega(k_1)\left(\frac{k_z}{k_{z_1}}\right)^{2\alpha+5}\left(\frac{k_\perp}{k_{\perp_1}}\right)^{2\beta+4}\right. \\ \left. - \Omega(k_2)\left(\frac{k_z}{k_{z_2}}\right)^{2\alpha+5}\left(\frac{k_\perp}{k_{\perp_2}}\right)^{2\beta+4}\right\}. \end{aligned}$$

This equation indicates that the expression in braces turns zero for $\alpha = -5/2$ and $\beta = -2$, so that the required solutions have the following form:

$$n_k = Ak_z^{-5/2}k_\perp^{-2}, \quad N_k = Bk_z^{-5/2}k_\perp^{-2} \tag{44}$$

These solutions correspond to the spectra with a constant energy flux P_ε. As above, a relationship between the coefficients A and B is determined from the stationary equations (32). These equations lead to the same estimate:

$$c_sA \sim V_AB.$$

The Kolmogorov type solutions obtained above refer only to the channel of interaction between the Alfvén waves and slow magnetoacoustic waves, which markedly reduces the significance of these results. It should be recalled that processes involving the fast magnetoacoustic waves are on the same order of magnitude and, hence, cannot be ignored. However, this channel can be successfully included into the above scheme without need in significant improvements. As was noted in the preceding section, a maximum scattering of the A-waves (i.e., of the Alfvén waves and fast magnetoacoustic waves) takes place for the maximum x-projection of the momentum transferred to the slow acoustic waves. It is naturally assumed that this behavior of the A-wave scattering amplitude would lead to strongly anisotropic distributions of the waves concentrated in the k-space within a narrow-angle cone in the magnetic field direction: $k_z \gg k_\perp$. Under these conditions, we may assume that the fast sound frequency coincides with that of the Alfvén waves:

$$\omega_2 \approx |k_z|V_A.$$

Another important circumstance is that the matrix element of the interaction between A-weaves and the slow magnetoacoustic waves is diagonal with respect to the polarization λ:

$$V^{\lambda\lambda_1}_{k'k_1'k_2} \approx \delta_{\lambda\lambda_1}V^{11}_{k'k_1'k_2}.$$

Thus, in the case of an almost longitudinal (i.e., extended in the magnetic field direction) distribution, we observe no difference between the Alfvén waves and fast magnetoacoustic waves. Moreover, there is even no energy exchange between these waves. This implies that (in the given angular range) the Kolmogorov spectra for the fast magnetoacoustic waves will exhibit the same shape as the spectra (41) and (44) obtained above. In these expressions, we have to replace N_k and B by N_k^λ and B_λ and determine the coefficient A as

$$A \sim \beta_{-1/2} \frac{\sum B_\lambda^2}{\sum B_\lambda}.$$

The Kolmogorov spectra (41) and (44) possess a physical meaning provided that the condition of local turbulence is fulfilled, according to which the contribution to the wave interaction due to the regions of pumping and damping must be small. This condition reduces to the requirement of convergence of the integrals in Eqs. (36) and (37).

The convergence of integrals with respect to $k_z$in the kinetic equations is ensured by the presence of two δ-functions of k_z. The integrals over transverse wavevectors exhibit logarithmic divergence. In our opinion, the logarithmic divergence is not as dangerous as the power-type behavior, since it falls on the boundary of locality. The appearance of this divergence is related to the aforementioned bihomogeneity of the probability U. If the medium is isotropic and the matrix elements of V and the frequencies possess the same degree of homogeneity as that in the MHD case of $\beta \ll 1$ (this situation takes place for the Mandel'shtam–Brillouin scattering), the condition of locality would be satisfied (see [34]). The condition of bihomogeneity for the interaction of Alfvén waves and slow magnetoacoustic waves is violated in the case of a nearly transverse wave propagation:

$$k_\perp / k_z \sim \beta_{-1/2},$$

ad for the interaction of fast and slow magnetoacoustic waves, in the region of angles

$$k_\perp / k_z \sim \beta_{1/2}.$$

For this reason, the integrals in the kinetic equations should be truncated at smaller angles ($\lesssim \beta_{1/2}$). The logarithmic divergence can be eliminated by introducing powers for logarithms of the transverse momenta $k_\perp$ in the spectra (41) and (44). However, this procedure does not lead to determining these powers (while ensuring the convergence of integrals). Finally, it should be noted that both spectra (41) and (44) are characterized by the same dependence on the transverse momenta:

$$n_k, N_k \propto k_\perp^{-2},$$

which coincides with the degree of homogeneity $\delta(\mathbf{k}_\perp)$. This implies that, besides the anisotropic Kolmogorov spectra (41) and (44), there may exist singular Kolmogorov spectra of the type

$$n_k = A k_z^{-2} \delta(k_\perp), \quad N_k = B k_z^{-2} \delta(k_\perp)$$

and

$$n_k = A k_z^{-5/2} \delta(\mathbf{k}_\perp), \quad N_k = B k_z^{-5/2} \delta(\mathbf{k}_\perp).$$

A rigorous answer to the question as to which stationary spectra are in fact realized can be obtained from the investigation of stability of the obtained solutions or (on a qualitative level) from the results of numerical modeling. Both approaches require special consideration.

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SPELL: automodel, tis, Oraevskii, curlS, ht, bihomogeneity