Compressible dynamics of magnetic field lines for incompressible magnetohydrodynamic flows

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It is demonstrated that the deformation of magnetic field lines in incompressible magnetohydrodynamic flows results from a compressible mapping associated with the transverse motion of fluid particles. Appearance of zeros for the Jacobian of this mapping corresponds to the breaking of magnetic field lines and the local blowup of the magnetic field intensity. The occurrence of such events is found to be unlikely in two dimensions but possible in three dimensions. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1669392]

I. INTRODUCTION

An important property of ideal magnetohydrodynamics (MHD) is the frozenness of the magnetic field in the plasma: fluid particles remain pasted on their magnetic lines, which are driven by the transverse velocity component. This property enables one to provide a global description of the dynamics of the magnetic field lines and to conjecture the appearance of a new kind of singularities for three-dimensional ideal MHD flows. These events are associated with the local blowup of the magnetic field intensity as the result of two magnetic field lines getting into contact. This magnetic field line frozen state is indeed the starting point for the development of a mixed Lagrangian–Eulerian description of ideal MHD flows, named magnetic line representation (MLR) and first formulated in Ref. 1. The idea originates from the vortex line representation (VLR) of hydrodynamic flows[2] that involves a two-dimensional Lagrangian marker labeling each vortex line, together with a parametrization of these lines. In three dimensions (3D), this representation enables one to partially integrate the Euler equations with respect to a continuous infinity of integrals of motion called the Cauchy invariants. A main peculiarity of the transformation associated with the vortex line dynamics is its compressible character which, as recently pointed out by one of the authors,[3] is amenable to a simple interpretation. The Euler equations can be rewritten as the equations of motion for a charged *compressible* fluid moving under the action of effective self-consistent electric and magnetic fields satisfying Maxwell equations. The new velocity coincides with the velocity component transverse to vorticity, which, due to the frozen state property, identifies with the vortex line velocity. It is well known that the appearance of singularities in compressible flows is connected with the emergence of shocks, corresponding to the formation of folds in the classical catastrophe theory.[4] In the gas-dynamic case, this process is completely characterized by the mapping defined by the transition from the usual Eulerian to the Lagrangian description. A zero of the Jacobian corresponds to the emergence of a singularity for the spatial derivatives of the velocity and density of the fluid. Due to the compressible character of VLR, the phenomenon of breaking also becomes possible for vortex lines in ideal incompressible fluids. Vortex-line breaking was first studied for three-dimensional integrable hydrodynamics with Hamiltonian $\mathcal{H}=\int|\mathbf{\Omega}|d\mathbf{r}$, where $\mathbf{\Omega}$ is the vorticity.[5] This model and the Euler equation are both incompressible and have the same symplectic operator defining the Poisson structure. Breaking of vortex lines is associated with the touching of two vortex lines and results in an infinite vorticity. Recent numerical simulations[6,7] have suggested the possibility of such a scenario for the 3D Euler equation, but further investigations are required to reach a definite conclusion. In ideal MHD, we can expect the same behavior to hold for the magnetic field which is a frozen-in quantity. In two dimensions (2D), however, the fact that vorticity is perpendicular to the flow plane while the magnetic field lies in it puts a limit to the analogy, making magnetic field line breaking *a priori* possible in two dimensions, while singularities are excluded in 2D Euler flows. It will nevertheless be argued in this paper that magnetic field blowup is unlikely in 2D MHD.

In Sec. II, we recall the Cauchy formula for MHD flows, which plays a central role in the derivation of the Weber-type transformation discussed in Sec. III. This transformation is obtained by extending ideas of Ref. 3 to ideal incompressible MHD flows. We in particular indicate how the MHD equations can be partially integrated. Section IV addresses the two-dimensional case where two conservation laws are established. In Sec. V, we discuss the possibility of magnetic line breaking as a local blowup of the magnetic field, a process different from the gradient singularity associated with current sheets formation (Ref. 8 and references therein). A brief conclusion is provided by Sec. VI.

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II. CAUCHY FORMULA IN MHD

As is well known, the magnetic field $\mathbf{h}$ in ideal incompressible MHD obeys

$$\mathbf{h}_t = \mathrm{curl}(\mathbf{v}\times\mathbf{h}), \quad \mathrm{div}\,\mathbf{v}=0, \tag{1}$$

that formally coincides with the equation governing the vorticity $\mathbf{\Omega}$ in Euler hydrodynamics. Since only the transverse velocity $\mathbf{v}_\perp$ to the local magnetic field is relevant in this equation, we introduce new Lagrangian trajectories

$$\mathbf{r}=\mathbf{r}(\mathbf{a},t), \tag{2}$$

defined by

$$\frac{d\mathbf{r}}{dt}=\mathbf{v}_\perp(\mathbf{r},t), \tag{3}$$

$$\mathbf{r}|_{t=0}=\mathbf{a}. \tag{4}$$

Because the magnetic field is a frozen-in quantity, Eq. (3) simultaneously is the equation of motion for magnetic field lines.

It is easily established that the Jacoby matrix (of element $\hat{J}_{ij}=\partial x_j/\partial a_i$) obeys

$$\frac{d}{dt}\hat{J}=\hat{J}U, \tag{5}$$

where the matrix U has elements $U_{ij}=\partial v_{\perp j}/\partial x_i$. One then obtains the equations for the Jacobian $J=\det\hat{J}$ and for the inverse matrix $\hat{J}^{-1}$ with elements $\partial a_j/\partial x_i$ [where $\mathbf{a}=\mathbf{a}(\mathbf{r},t)$ is the inverse of the mapping defined in (2)], in the form

$$\frac{d}{dt}J=\mathrm{div}\,\mathbf{v}_\perp J, \tag{6}$$

and

$$\frac{d}{dt}\hat{J}^{-1}=-U\hat{J}^{-1}. \tag{7}$$

Since $\mathrm{div}\,\mathbf{v}_\perp$ is generically nonzero, the mapping (2) is compressible and the Jacobian J can vanish. This observation is central in the discussion of the possibility of magnetic field blowup presented in Sec. V.

By means of Eqs. (6) and (7), Eq. (1) is transformed into

$$D_t\left(Jh_i\frac{\partial a_j}{\partial x_i}\right)=0, \tag{8}$$

where $D_t=\partial_t+(\mathbf{v}_\perp\cdot\nabla)$ identifies with the material derivative d/dt used in (3). Integration of this equation leads to a "new" vector Lagrangian invariant

$$I_j(\mathbf{a})\equiv Jh_i\frac{\partial a_j}{\partial x_i}, \tag{9}$$

that coincides with the initial magnetic field $\mathbf{h}_0(\mathbf{a})$ and is the analog of the Cauchy invariants for ideal hydrodynamics. The magnetic field $\mathbf{h}$ is then given by

$$\mathbf{h}(\mathbf{r},t)=\frac{(\mathbf{h}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}})\mathbf{r}(\mathbf{a},t)}{J}. \tag{10}$$

III. WEBER-TYPE TRANSFORMATION

Equation (10) is the basis of the magnetic line representation.[1] Another important formula for MLR follows from the velocity equation

$$\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p+\mathrm{curl}\;\mathbf{h}\times\mathbf{h}, \tag{11}$$

where we normalized the magnetic field by the factor $\sqrt{4\pi\rho}$ (where ρ is the density) so that $\mathbf{h}$ has the dimension of a velocity.

We also decompose the velocity $\mathbf{v}=\mathbf{v}_\perp+\mathbf{v}_\tau$ into transverse and tangential components and substitute in Eq. (11), which is rewritten as

$$\partial_t\mathbf{v}_\perp+(\mathbf{v}_\perp\cdot\nabla)\mathbf{v}_\perp=\mathbf{E}+\mathbf{v}_\perp\times\mathbf{H}+\mathbf{F}^{\mathrm{mhd}}, \tag{12}$$

where we introduced new effective "electric" and "magnetic" fields

$$\mathbf{E}=-\nabla\left(p+\frac{v_\tau^2}{2}\right)-\frac{\partial\mathbf{v}_\tau}{\partial t}, \tag{13}$$

$$\mathbf{H}=\mathrm{rot}\;\mathbf{v}_\tau. \tag{14}$$

In Eq. (12), the force $\mathbf{F}^{\mathrm{mhd}}=\mathbf{j}\times\mathbf{h}$, involves the renormalized current

$$\mathbf{j}=\mathrm{curl}\,\mathbf{h}-(\mathbf{v}\cdot\mathbf{h})/h^2\mathrm{curl}\;\mathbf{v}. \tag{15}$$

As seen from (13) and (14), the new auxiliary electric and magnetic fields can be expressed in terms of scalar and vector potentials $\varphi=p+(\mathbf{v}_\tau^2/2)$ and $\mathbf{A}=\mathbf{v}_\tau$, so that the two Maxwell equations

$$\mathrm{div}\,\mathbf{H}=0, \quad \frac{\partial\mathbf{H}}{\partial t}=-\mathrm{curl}\;\mathbf{E}$$

are automatically satisfied. In this case, the vector potential $\mathbf{A}$ has the gauge

$$\mathrm{div}\,\mathbf{A}=-\mathrm{div}\,\mathbf{v}_\perp,$$

which is equivalent to the incompressibility condition $\mathrm{div}\,\mathbf{v}=0$.

The two other Maxwell equations define auxiliary charge density and current, which follow from relations (13) and (14).

New terms in the right-hand side of Eq. (12) also have a mechanical interpretation. The Lorentz force $\mathbf{v}_\perp\times\mathbf{H}$ plays the role of a Coriolis force. The potential φ has a direct connection with the Bernoulli formula. The term $\partial_t\mathbf{v}_\tau$ results from the noninertial character of the coordinate system.

In Eq. (12), we make the change of variable defined by mapping (2). As a result, the equations of motion are expressed in a quasi-Hamiltonian form, analogous to Eq. (20) of Ref. 3

$$D_t\mathbf{P}=-\frac{\partial h}{\partial\mathbf{r}}+\mathbf{F}^{\mathrm{mhd}}, \quad D_t\mathbf{r}=\frac{\partial h}{\partial\mathbf{P}}, \tag{16}$$

where the Hamiltonian h is given by the standard expression

$$h=\frac{1}{2}(\mathbf{P}-\mathbf{A})^2+\varphi\equiv p+\frac{\mathbf{v}^2}{2},$$

in terms of the generalized momentum $\mathbf{P}=\mathbf{v}_\perp+\mathbf{A}$ (that identifies with $\mathbf{v}$), and thus coincides with the Bernoulli "invariant" for a nonmagnetic fluid. The first equation of the system (16) contains an addition term $\mathbf{F}^{\mathrm{mhd}}$ and therefore we qualify (16) as quasi-Hamiltonian.

Introducing a new vector

$$u_k=P_i\frac{\partial x_i}{\partial a_k},$$

depending on t and $\mathbf{a}$, one easily obtains from (16) that this vector obeys

$$D_t u_k=\frac{\partial}{\partial a_k}\left(-p+\frac{v_\perp^2}{2}-\frac{v_\tau^2}{2}\right)+F_i^{\mathrm{mhd}}\frac{\partial x_i}{\partial a_k}. \tag{17}$$

Using (10) and the identity

$$\epsilon_{\alpha\beta\gamma}\frac{\partial x_i}{\partial a_\beta}\frac{\partial x_j}{\partial a_\gamma}=\epsilon_{ijk}J\frac{\partial a_\alpha}{\partial x_k}, \tag{18}$$

one has

$$F_i^{\mathrm{mhd}}\frac{\partial x_i}{\partial \mathbf{a}}=\mathbf{h}_0(\mathbf{a})\times\mathbf{S},$$

where

$$\mathbf{S}=(\mathbf{j}\cdot\nabla_{\mathbf{r}})\mathbf{a}.$$

Equation (17) is thus rewritten as

$$D_t\mathbf{u}=\nabla_{\mathbf{a}}\left(-p+\frac{v_\perp^2}{2}-\frac{v_\tau^2}{2}\right)+\mathbf{h}_0(\mathbf{a})\times\mathbf{S}. \tag{19}$$

Integrating in time then leads to the Weber-type transformation

$$\mathbf{u}=\mathbf{u}_0(\mathbf{a})+\nabla_{\mathbf{a}}\Phi+h_0(\mathbf{a})\times\mathbf{W}, \tag{20}$$

where the potential Φ satisfies a Bernoulli-type equation

$$D_t\Phi=-p+\frac{v_\perp^2}{2}-\frac{v_\tau^2}{2},$$

and the vector $\mathbf{W}$ obeys

$$D_t\mathbf{W}=\mathbf{S}. \tag{21}$$

If initially $\Phi|_{t=0}=0$ and $\mathbf{W}|_{t=0}=0$, the integration "constant" $\mathbf{u}_0(\mathbf{a})$ coincides with the initial velocity $\mathbf{v}_0(\mathbf{a})$. This vector $\mathbf{u}_0(\mathbf{a})$ is thus a new Lagrangian invariant.

To get a closed description we eliminate the pressure p by applying the curl operator (with respect to $\mathbf{a}$ variables) on Eq. (20)

$$\mathrm{curl}_{\mathbf{a}}\ \mathbf{u}=\mathrm{curl}_{\mathbf{a}}\ \mathbf{u}_0(\mathbf{a})+\mathrm{curl}_{\mathbf{a}}[\mathbf{h}_0(\mathbf{a})\times\mathbf{W}]. \tag{22}$$

This equation can also be rewritten as

$$\boldsymbol{\Omega}(\mathbf{r},t)=\frac{(\boldsymbol{\Omega}_0(\mathbf{a}),t)\cdot\nabla_{\mathbf{a}})\mathbf{r}(\mathbf{a},t)}{J}. \tag{23}$$

Here, $\boldsymbol{\Omega}_0(a,t)$ is given by

$$\boldsymbol{\Omega}_0(\mathbf{a},t)=\boldsymbol{\Omega}_0(\mathbf{a})+\mathrm{curl}_{\mathbf{a}}[\mathbf{h}_0(\mathbf{a})\times\mathbf{W}],$$

where $\boldsymbol{\Omega}_0(\mathbf{a})$ is the initial vorticity. When $\mathbf{h}_0(\mathbf{a})=0$, Eq. (23) reduces to the Cauchy formula for vorticity in ideal hydrodynamics.

The vector $\mathbf{W}$ is determined from Eq. (21), which is rewritten as

$$D_t\mathbf{W}=\frac{(\mathbf{v}\cdot\mathbf{b})}{b^2}\boldsymbol{\Omega}_0(\mathbf{a},t)-\frac{1}{J}\mathrm{curl}_a\left(\frac{\hat{g}\mathbf{h}_0(\mathbf{a})}{J}\right), \tag{24}$$

where $\hat{g}$ is the MLR metric tensor defined by

$$g_{\alpha\beta}=\frac{\partial x_i}{\partial a_\alpha}\cdot\frac{\partial x_i}{\partial a_\beta},$$

and $\mathbf{b}=J\mathbf{h}$ is given by Eq. (10).

As a result, the system formed by Eq. (24) for the vector $\mathbf{W}$, Eqs. (3)–(4) defining the mapping, Eqs. (10) and (23) defining the magnetic field and the vorticity, together with Eqs. (10), (23) and the relation between velocity and vorticity

$$\boldsymbol{\Omega}=\mathrm{curl}_r\mathbf{v},\quad \mathrm{div}_r\,\mathbf{v}=0, \tag{25}$$

provides a closed description of the dynamics of a magnetic line in incompressible MHD (to be compared with Ref. 1). These equations are solved with respect to two Lagrangian invariants $\mathbf{h}_0(\mathbf{a})$ and $\boldsymbol{\Omega}_0(\mathbf{a})$. It is possible to show[1] that conservation of these invariants in MHD is a consequence of relabeling symmetry, as it is the case for Euler equation (see, e.g., the reviews in Refs. 9 and 10).

The magnetic line representation involving the local change of variables $\mathbf{r}=\mathbf{r}(\mathbf{a},t)$, breaks down at singular points where the Jacobian is zero or infinity and the normal velocity is not defined.

Let us consider the null point $\mathbf{r}=\mathbf{r}(t)$ defined by

$$\mathbf{h}(\mathbf{r}(t),t)=0. \tag{26}$$

Differentiating this equation with respect to time, we get

$$\frac{\partial\mathbf{h}}{\partial t}+(\dot{\mathbf{r}}(t)\cdot\nabla)\mathbf{h}=0,$$

with $\dot{\mathbf{r}}(t)=\mathbf{v}(\mathbf{r}(t),t)$, which shows that the null points are advected by the flow. The velocity $\mathbf{v}$ at these points is defined by inverting the curl operator in (25).

Null points are topological singularities for the tangent vector field $\tau(\mathbf{r})$. Their classification depends on the space dimension D. Topological constraints that can be considered as additional conditions for the MLR system can be written as integrals of the vector field $\tau(\mathbf{r})$ and its derivatives over the boundary of simply connected regions (in 3D) or along closed contours (in 2D) enclosing the null points. In $D=2$, one has

$$\oint(\nabla\varphi\cdot d\mathbf{r})=2\pi m, \tag{27}$$

where φ is the angle between the vector $\boldsymbol{\tau}$ and the x axis and m is an integer often called topological charge. It is equal to the total number of turns of the vector τ while passing around the closed contour encircling the null point. For instance, for X points or O points, $m=\pm 1$.

In $D=3$, the topological charge is defined as the degree of the mapping $\mathcal{S}^2\to\mathcal{S}^2$, given by

$$\int_{\partial V}\epsilon_{\alpha\beta\gamma}\ (\tau\cdot[\partial_\beta\tau\times\partial_\gamma\tau])dS_\alpha=4\pi m, \tag{28}$$

where the integration is performed over the boundary ∂V of a region V containing null points.

Conditions (25)–(28) complete the MLR equations in the general case when the Jacobian has localized zeros.

The above representation that involves simultaneous use of Lagrangian variables in Eqs. (3), (24), (10), (23) and Eulerian ones in (25), makes the numerical integration very cumbersome. It is therefore of interest to look for a representation formulated in the sole physical space.

Let us consider the inverse of the mapping $\mathbf{a}=\mathbf{a}(\mathbf{r},t)$. Using Eq. (3), one has

$$\partial_t \mathbf{a} + (\mathbf{v}_\perp \cdot \nabla)\mathbf{a} = 0. \tag{29}$$

From (18), Eq. (10) for the magnetic field is rewritten as

$$\mathbf{h} = \tfrac{1}{2}\epsilon_{ijk} h_{0i}(\mathbf{a})[\nabla a_j \times \nabla a_k]. \tag{30}$$

Formula (23) for the vorticity in $\mathbf{r}$ variable becomes

$$\mathbf{\Omega}(\mathbf{r},t) = \mathrm{curl}(V_i \nabla a_i), \tag{31}$$

where

$$\mathbf{V} = \mathbf{v}_0(\mathbf{a}) + \mathbf{h}_0(\mathbf{a}) \times \mathbf{W}.$$

Similarly, Eq. (21) for the vector $\mathbf{W}$ transforms into

$$\partial_t \mathbf{W} + (\mathbf{v}_\perp \cdot \nabla)\mathbf{W} = -(\mathbf{j}\cdot\nabla)\mathbf{a}, \tag{32}$$

with initial condition $\mathbf{W}|_{t=0}=0$. Here, the generalized current $\mathbf{j}$ is given by (15).

These equations are completed by relation (25) and the definition of the normal velocity $\mathbf{v}_\perp = \hat{\Pi}\ \mathbf{v}$, where the projector $\hat{\Pi}$ is defined by means of the unit tangent vector $\tau = \mathbf{h}/h$ as $\Pi_{\alpha\beta} = \delta_{\alpha\beta} - \tau_\alpha \tau_\beta$. They provide a closed system for ideal MHD flows, where all the spatial derivatives are taken with respect to $\mathbf{r}$ variables.

IV. CONSERVATION LAWS IN TWO DIMENSIONS

The magnetic line representation significantly simplifies in two dimensions where the magnetic field lies on the same plane as the flow. It is convenient to introduce, instead of the initial position $\mathbf{a}$, the scalar magnetic potential ψ defined by

$$h_x = \frac{\partial \psi}{\partial y}, \quad h_y = -\frac{\partial \psi}{\partial x},$$

and a Cartesian coordinate y.

By fixing ψ, we select a magnetic line given by

$$\frac{dx}{\partial\psi/\partial y} = -\frac{dy}{\partial\psi/\partial x}.$$

The difference $\psi_1 - \psi_2$ is equal to the flux of magnetic field between two lines with different values of ψ.

In 2D, ψ is a Lagrangian invariant, since it follows from the integration of the induction equation (1) that

$$\frac{\partial \psi}{\partial t} + (\mathbf{v}\cdot\nabla)\quad \psi = 0. \tag{33}$$

The potential

$$\psi = \psi(x,y,t), \tag{34}$$

can then be taken as a Lagrangian marker of the magnetic lines. Solving Eq. (34) locally in the form $y = y(x,\psi,t)$ provides the desired mapping that replaces (2).

This change of variables, being a mixed Lagrangian–Eulerian one, realizes a transformation to a *curvilinear* system of coordinates movable with magnetic lines. In order to implement the transformation from variables (x,y,t) to (x,ψ,t) in Eqs. (33) and (11), we use

$$\frac{\partial f}{\partial t} = \frac{1}{y_\psi}[f_t y_\psi - f_\psi y_t], \tag{35}$$

$$\frac{\partial f}{\partial x} = \frac{1}{y_\psi}[f_x y_\psi - f_\psi y_x], \tag{36}$$

$$\frac{\partial f}{\partial y} = \frac{f_\psi}{y_\psi}, \tag{37}$$

where derivatives are taken relatively to (x,y,t) in the left-hand sides of the above equations and to (x,ψ,t) in the right-hand sides.

Equation (34) for the magnetic potential then transforms into an equation for the magnetic line ψ

$$y_t + v_x y_x = v_y. \tag{38}$$

This equation is a kinematic condition. As the equation of motion (3), the dynamics of y is prescribed by the velocity component normal to the magnetic field line $y_t = v_\perp \sqrt{1+y_x^2}$, where $v_\perp = (\mathbf{v}\cdot\mathbf{n})$ and $\mathbf{n} = (1/\sqrt{1+y_x^2})(-y_x,1)$. In terms of the new variables, the magnetic field is given by

$$h_x = \frac{1}{y_\psi}, \quad h_y = \frac{y_x}{y_\psi},$$

which are equivalent to the Cauchy formula (10) for the magnetic field in 2D. The derivative y_ψ in the denominators holds for the Jacobian J. The equation for the quantity y_ψ can be found by differentiating (38) with respect to ψ and applying the incompressibility condition in the form

$$\frac{\partial v_x}{\partial x} y_\psi - \frac{\partial v_x}{\partial \psi} y_x + \frac{\partial v_y}{\partial \psi} = 0. \tag{39}$$

This results in a continuity equation for y_ψ

$$\partial_t y_\psi + \partial_x (v_x y_\psi) = 0, \tag{40}$$

so that y_ψ has the meaning of a layer density.

Another useful relation can be obtained from the equations for the velocity components v_x and v_y that now read

$$\partial_t v_x + v_x \partial_x v_x = -\partial_x p + (\partial_\psi p - j)\frac{y_x}{y_\psi}, \tag{41}$$

$$\partial_t v_y + v_x \partial_x v_y = -(\partial_\psi p - j)\frac{1}{y_\psi}, \tag{42}$$

where $j = \mathrm{curl}\ \mathbf{h}$ is the current directed along the z direction. It is then convenient to introduce

$$U = v_x + y_x v_y,$$

where y_x obeys the equation

$$\partial_t y_x + v_x \partial_x y_x + y_x \partial_x v_x = \partial_x v_y,$$

derived from (38). The function U coincides up to the factor $1/\sqrt{1+y_x^2}$ with the velocity component tangent to the magnetic field $v_\tau = (1/\sqrt{1+y_x^2})\,U$. One easily gets

$$\partial_t U + \partial_x (v_x U) = -\partial_x (p - v^2/2), \tag{43}$$

that can be viewed as a differential form of the Kelvin theorem.

Combination of Eqs. (40) and (42) implies that $w = v_y y_\psi$ obeys

$$\partial_t w + \partial_x (v_x w) = -\partial_\psi p + j. \tag{44}$$

To find the analog of (23) in the 2D case, it is convenient to make the change of variables $y = y(x,\psi,t)$ in the vorticity equation

$$\partial_t \Omega + (\mathbf{v}\cdot\nabla)\Omega = \nabla j \times \nabla \psi. \tag{45}$$

Substituting relations (35)–(37) into (45) and using Eq. (38), we get

$$\partial_t \Omega + v_x \partial_x \Omega_x = \frac{\partial_x j}{y_\psi}.$$

Equations (40) and (43) provide conservation laws for 2D incompressible MHD. They remain valid in the hydrodynamic limit, provided ψ is replaced by vorticity or by any other Lagrangian invariant.

V. POSSIBILITY OF MAGNETIC LINE BREAKING

An important property of the magnetic line representation concerns the compressibility of the mapping defined by (2), which permits magnetic line breaking. At the breaking point, the magnetic field, according to (10), becomes infinite due to the vanishing of the Jacobian. As it follows from Refs. 3, 5–7, the possibility of vortex line breaking depends on the space dimension. For two-dimensional flows described by the Euler equations, vorticity is perpendicular to the flow plane, and therefore $\operatorname{div}\mathbf{v}_\perp = 0$. As the consequence, the corresponding mapping is incompressible and the Jacobian remains constant.

For 2D incompressible MHD, the situation is different since the magnetic field lies in the flow plane. The velocity can therefore be decomposed into transverse and longitudinal components relative to *the magnetic field direction*. In such a case $\operatorname{div}\mathbf{v}_\perp \neq 0$ and the breaking of magnetic lines is not *a priori* excluded. Its actual occurrence is nevertheless dependent on space dimension.

Let us thus assume that a breaking of magnetic lines occurs. Denote by $t = \tilde{t}(\mathbf{a}) > 0$ the positive roots of the equation

$$J(\mathbf{a},t) = 0,$$

and find the minimal value $t_0 = \min_a \tilde{t}(\mathbf{a})$, which defines the first instant of time when the Jacobian vanishes. Let $\mathbf{a} = \mathbf{a}_0$ be the Lagrangian coordinate of the point where this minimum is attained. We first consider that near the singular point, as $t \to t_0$, the Jacobian behaves as

$$J = \alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j, \tag{46}$$

where $\alpha > 0$, γ_{ij} is a positive definite (generically nondegenerated) matrix and $\Delta\mathbf{a} = \mathbf{a} - \mathbf{a}_0$. This assumes that the magnetic field does not vanish at the collapse point and in particular that the three vectors $\partial\mathbf{r}/\partial a_i$ $(i = 1,2,3)$ lie in the same plane, with none of them vanishing. In this case, Eq. (10) is rewritten as

$$\mathbf{h} = \frac{\mathbf{b}}{\alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j}, \tag{47}$$

where $\mathbf{b} = (\mathbf{h}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}})\mathbf{r}|_{t_0,\mathbf{a}_0}$. This corresponds to a blowup of the magnetic field $\mathbf{h}(\mathbf{a}_0)$ like $1/(t_0 - t)$.

The MHD equations conserve the energy $\mathcal{E}$ given by the sum of the kinetic $\mathcal{E}_k = \int (\mathbf{v}^2/2)\, d\mathbf{r}$ and magnetic $\mathcal{E}_h = \int (\mathbf{h}^2/2)\, d\mathbf{r}$ energies, where both have to remain finite as $t \to t_0$.

Let us estimate the contribution to the magnetic energy originating from the neighborhood of a possible singularity (47)

$$\mathcal{E}_h \approx \int \frac{b^2}{J^2}\, d\mathbf{r}. \tag{48}$$

By changing variables from $\mathbf{r}$ to $\mathbf{a}$, the contribution to this integral arising from a ball centered in $\mathbf{a}_0$ and of radius $R \sim \tau^{1/2}$ where $\tau = t_0 - t$, is rewritten as

$$\mathcal{E}_h^s \approx b^2 \int \frac{d\mathbf{a}}{\alpha\tau + \gamma_{ij} a_i a_j} \propto (t_0 - t)^{(D-2)/2}. \tag{49}$$

The retained size of the ball is the largest compatible with the asymptotics. The contribution due to rest of the domain being most likely finite, we conclude that a magnetic field blowup in not excluded in 3D for the assumed expansion of the Jacobian. The same conclusion holds if the Jacobian vanishes like $(t_0 - t)^n$ at the singularity point, with a ball radius modified accordingly. At a point where the matrix γ is degenerated with, for example, one eigenvalue λ_1 being zero, the Jacobian locally becomes

$$J = \alpha(t_0 - t) + \tilde{\gamma}_{ij} a_i^\perp a_j^\perp + \beta a_1^4, \tag{50}$$

where $\mathbf{a}^\perp$ holds for the projection of the vector $\mathbf{a}$, transverse to the direction of the eigenvector associated with the zero eigenvalue. The contribution of the singularity to the magnetic energy then scales like $\mathcal{E}_h^s \sim (t_0 - t)^{1/4}$, a behavior which again does not contradict the possible existence of a singularity.

In $D = 2$, the conclusion can be different. Since the contribution of the selected ball to the magnetic energy does not tend to zero as $t \to t_0$, a small extension of this domain to a ball of size R can lead to a logarithmic divergence $\mathcal{E}_h^s \sim B^2 \log(\gamma R^2/\alpha\tau) \to \infty$. The divergence becomes more dramatic in the case of a degenerate matrix γ, for which $\mathcal{E}_h^s \sim (t_0 - t)^{-1/4}$. This observation leads us to conjecture that a blowup of the magnetic field is probably excluded in two dimensions but not necessary in three dimensions. Note that the conservations laws (40) and (43) for the two-dimensional problem derived in Sec. III, could possibly be useful for a rigorous proof of the absence of magnetic blowup.

VI. CONCLUSION

The mechanism for a finite-time singularity addressed in this paper corresponds to the breaking of magnetic field lines resulting in a catastrophic growth of the local amplitude of the magnetic field. It is worth noticing that this process does not contradict the necessary condition for blowup in MHD[11] that represents the analog of the Beale–Kato–Majda inequality.[12] According to Ref. 11, the velocity and magnetic field retain their smoothness on a time interval $[0,T]$ as long as the time integral of the supremum of the vorticity $|\boldsymbol{\Omega}(t)|_\infty$ and current $|\mathbf{j}(t)|_\infty$ obeys

$$\int_0^T (|\boldsymbol{\Omega}(t)|_\infty + |\mathbf{j}(t)|_\infty)\,dt < \infty.$$

Hence, a finite-time singularity of any kind must be accompanied by the blowup of $\boldsymbol{\Omega}$ and $\nabla \mathbf{h}$. However, this criterion does not exclude a blowup of the magnetic field amplitude as well. Constraints are nevertheless provided by regularity theorems; one result for example states that the solution remains globally smooth if the initial magnetic field has a mean component sufficiently large compared to the fluctuations, assumed to be localized.[13] This property is a consequence of the fact that only counterpropagating Alfvén wave packets interact nonlinearly.

A specific conclusion of this paper is that blowup of the magnetic field amplitude resulting from the breaking of magnetic field lines is unlikely in two space dimensions. Nevertheless, the present formalism cannot capture the behavior near a neutral X point where numerical evidence and self-similar reductions indicate that the current amplifies exponentially in time.[14,15]

Furthermore, recent direct numerical simulations of 3D MHD indicate the formation of quasi-two-dimensional current sheets that result in a depletion of the nonlinearity strength,[16] a mechanism that could prevent singularities. In order to validate the scenario of magnetic field intensity blowup discussed in this paper, it is thus of interest to look for initial conditions that do not lead to bidimensionalization and correspond to an initial velocity field whose component transverse to the local magnetic field has a significant divergence.

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