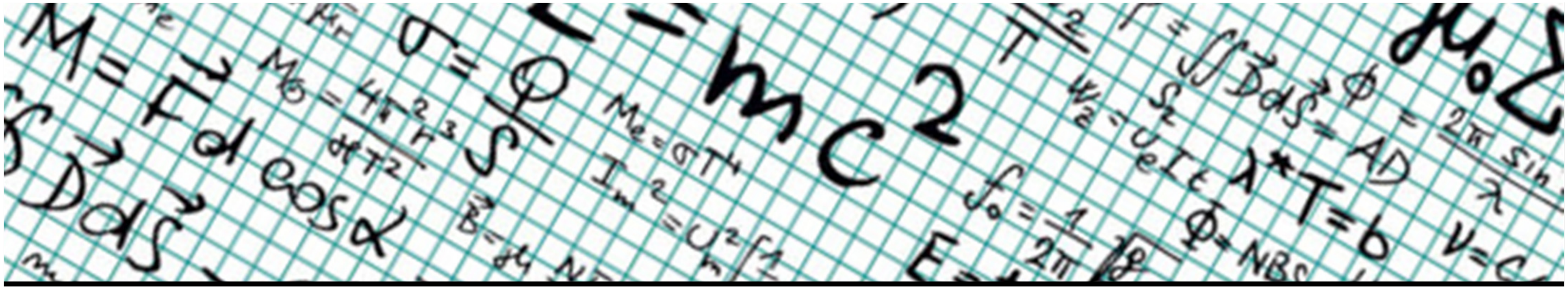

Formulation of multi-fluid species equations

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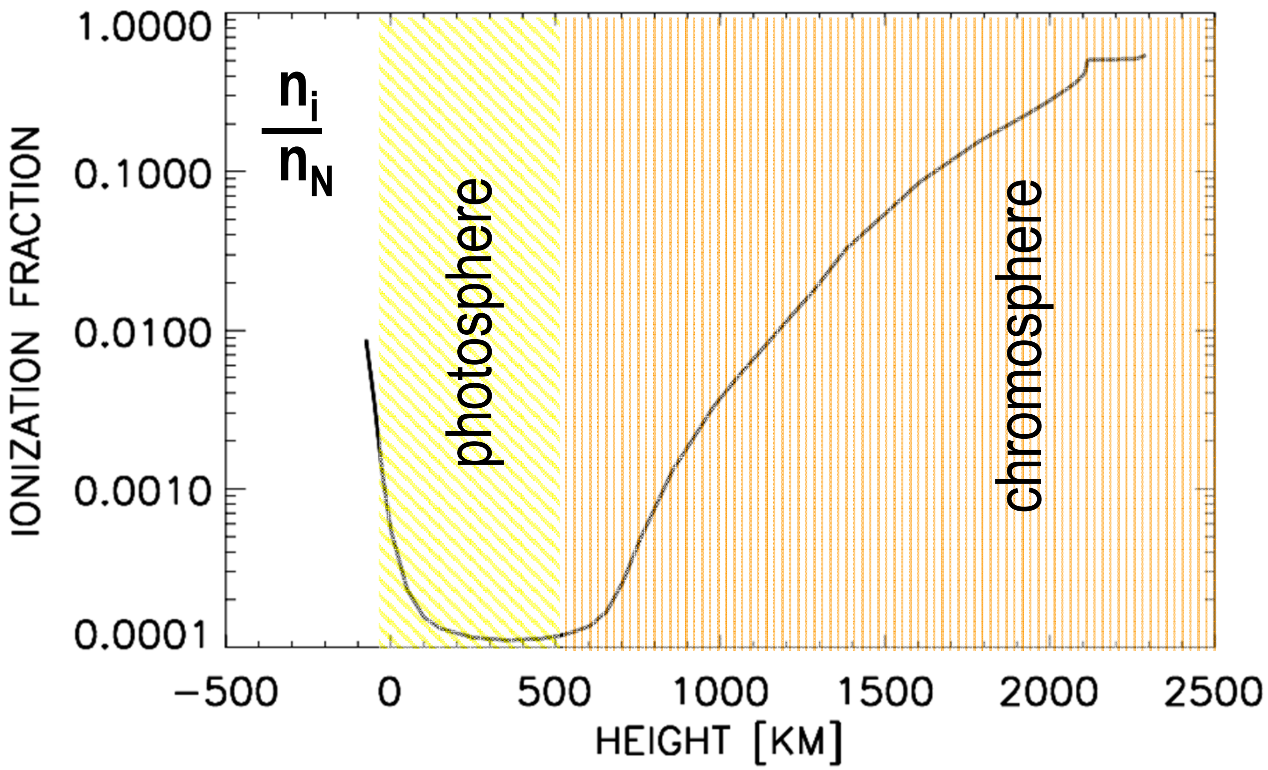
Degree of Ionization in VAL-C model



Equations for individual species

Species: Particle density

- $n_\alpha = \sum_I n_{\alpha \mathrm{I}} = \sum_I \sum_E n_{\alpha \mathrm{IE}}$

α≡ species ; I ≡ ionisation state; E ≡ excitation state

- $m_{\alpha \mathrm{IE}} = m_{\alpha \mathrm{I}} = m_\alpha$
- $\vec{u}_{\alpha \mathrm{IE}} = \vec{u}_{\alpha \mathrm{I}} \neq \vec{u}_\alpha$

- $n_e = \sum_\alpha \sum_{I \geq 1} I \cdot n_{\alpha \mathrm{I}}$

Equations for individual species

Boltzmann equation

$$\frac{\partial f_{\alpha \mathrm{IE}}}{\partial t} + \vec{v}\vec{\nabla} f_{\alpha \mathrm{IE}} + \vec{a}\vec{\nabla}_v f_{\alpha \mathrm{IE}} = \left(\frac{\partial f_{\alpha \mathrm{IE}}}{\partial t}\right)_{\mathrm{coll}}$$

$$\vec{v} = \vec{u}_{\alpha \mathrm{IE}} + \vec{c}_{\alpha \mathrm{IE}} = \vec{u}_{\alpha \mathrm{I}} + \vec{c}_{\alpha \mathrm{I}}$$

Equations for individual species

Transport equations

$$\frac{\partial}{\partial t}(n_{\alpha\mathrm{IE}}\langle\chi\rangle_{\alpha\mathrm{IE}}) + \vec{\nabla}(n_{\alpha\mathrm{IE}}\langle\chi\vec{v}\rangle_{\alpha\mathrm{IE}}) - n_{\alpha\mathrm{IE}}\langle\vec{a}\vec{\nabla}_v\chi\rangle_{\alpha\mathrm{IE}} = \int_V \chi\left(\frac{\partial f_{\alpha\mathrm{IE}}}{\partial t}\right)_{\mathrm{coll}} d^3v$$

$$\frac{\partial}{\partial t}(n_{\alpha\mathrm{IE}}\langle\vec{\chi}\rangle_{\alpha\mathrm{IE}}) + \vec{\nabla}(n_{\alpha\mathrm{IE}}\langle\vec{\chi}\otimes\vec{v}\rangle_{\alpha\mathrm{IE}}) - n_{\alpha\mathrm{IE}}\langle(\vec{a}\vec{\nabla}_v)\vec{\chi}\rangle_{\alpha\mathrm{IE}} = \int_V \vec{\chi}\left(\frac{\partial f_{\alpha\mathrm{IE}}}{\partial t}\right)_{\mathrm{coll}} d^3v$$

Mass conservation

- $\chi = m_{\alpha\mathrm{IE}} = m_\alpha$
- $\langle\chi\rangle_{\alpha\mathrm{IE}} = m_{\alpha\mathrm{IE}}$
- $\langle\chi\vec{v}\rangle_{\alpha\mathrm{IE}} = m_{\alpha\mathrm{IE}}\langle\vec{v}_{\alpha\mathrm{IE}}\rangle = m_{\alpha\mathrm{IE}}\vec{u}_{\alpha\mathrm{IE}}$
- $\vec{\nabla}_v\chi = 0$

$$\frac{\partial\rho_{\alpha\mathrm{IE}}}{\partial t} + \vec{\nabla}(\rho_{\alpha\mathrm{IE}}\vec{u}_{\alpha\mathrm{I}}) = m_\alpha\int_V\left(\frac{\partial f_{\alpha\mathrm{IE}}}{\partial t}\right)_{\mathrm{coll}} d^3v = S_{\alpha\mathrm{IE}}$$

$$S_{\alpha\mathrm{IE}} = m_\alpha\sum_{\mathrm{I'E'}\neq\mathrm{IE}}(n_{\alpha\mathrm{I'E'}}P_{\alpha\mathrm{I'E'IE}} - n_{\alpha\mathrm{IE}}P_{\alpha\mathrm{IEI'E'}})$$

Inelastic collisions

photons

particles

$$P_{\alpha\mathrm{IEI'E'}} = F_{\alpha\mathrm{IEI'E'}} + C_{\alpha\mathrm{IEI'E'}}$$

Mass conservation

Electrons

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla}(\rho_e \vec{u}_e) = m_e \sum_{\alpha} \sum_{\mathrm{I}} \sum_{\mathrm{I}' \neq \mathrm{I}} (n_{\alpha \mathrm{I}'} P_{\alpha \mathrm{I}' \mathrm{I}} - n_{\alpha \mathrm{I}} P_{\alpha \mathrm{I} \mathrm{I}'})$$

Momentum conservation

- $\vec{\chi} = m_{\alpha \mathrm{IE}} \vec{v}$

$$\rho_{\alpha \mathrm{IE}} \frac{D\vec{u}_{\alpha \mathrm{I}}}{Dt} = n_{\alpha \mathrm{IE}} q_{\alpha \mathrm{I}} (\vec{E} + \vec{u}_{\alpha \mathrm{I}} \times \vec{B}) + \rho_{\alpha \mathrm{IE}} \vec{g} - \vec{\nabla} \hat{\mathbf{p}}_{\alpha \mathrm{IE}} + \vec{R}_{\alpha \mathrm{IE}} - \vec{u}_{\alpha \mathrm{I}} S_{\alpha \mathrm{IE}}$$

Elastic collisions

$$\rho_e \frac{D\vec{u}_e}{Dt} = -e n_e (\vec{E} + \vec{u}_e \times \vec{B}) + \rho_e \vec{g} - \vec{\nabla} \hat{\mathbf{p}}_{\mathbf{e}} + \vec{R}_e - \vec{u}_e S_e$$

Energy conservation

- $\chi = m_\alpha v^2/2 + E_{\alpha \mathrm{IE}}$
- $e_{\alpha \mathrm{IE}} = \frac{3}{2} p_{\alpha \mathrm{IE}} + n_{\alpha \mathrm{IE}} E_{\alpha \mathrm{IE}}$

$$\frac{De_{\alpha \mathrm{IE}}}{Dt} + e_{\alpha \mathrm{IE}} \vec{\nabla} \vec{u}_{\alpha \mathrm{I}} + \hat{\mathbf{p}}_{\alpha \mathrm{IE}} \vec{\nabla} \vec{u}_{\alpha \mathrm{I}} + \vec{\nabla} \vec{q}_{\alpha \mathrm{IE}} = Q_{\alpha \mathrm{IE}}$$

$$Q_{\alpha \mathrm{IE}} = M_{\alpha \mathrm{IE}} - \vec{u}_{\alpha \mathrm{I}} \vec{R}_{\alpha \mathrm{IE}} + \left(\frac{1}{2} u_{\alpha \mathrm{I}}^2 + E_{\alpha \mathrm{IE}}/m_\alpha \right) S_{\alpha \mathrm{IE}}$$

$$\frac{De_e}{Dt} + e_e \vec{\nabla} \vec{u}_e + \hat{\mathbf{p}}_{\mathbf{e}} \vec{\nabla} \vec{u}_e + \vec{\nabla} \vec{q}_e = M_e - \vec{u}_e \vec{R}_e + \frac{1}{2} u_e^2 S_e$$

Photons: Radiative Transfer equation

Boltzmann equation for photons

$$\frac{\partial f_R}{\partial t} + \vec{v}\vec{\nabla} f_R + \vec{F}\vec{\nabla}_p f_R = \left(\frac{\partial f_R}{\partial t}\right)_{\text{coll}}$$

$$\left.\begin{aligned} f_R &= f_R(\vec{r}, \vec{n}, \nu, t) \\ \vec{F} &= 0 \\ I_\nu(\vec{r}, \vec{n}, t) &= ch\nu f_R(\vec{r}, \vec{n}, \nu, t) \end{aligned}\right\} \quad \frac{1}{ch\nu}\left[\cancel{\frac{\partial I_\nu}{\partial t}} + c\vec{n}\vec{\nabla} I_\nu\right] = \frac{j_\nu - k_\nu I_\nu}{h\nu}$$

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu$$

Radiative Transfer equation

Photons: Radiative Transfer equation

Transport equations

$$\cancel{\frac{1}{c^2}\frac{\partial \vec{F}_R}{\partial t}} + \vec{\nabla}\hat{P}_R = \frac{1}{c}\int_0^\infty \oint (j_\nu - k_\nu I_\nu)\vec{n}\,d\omega d\nu$$

$$\cancel{\frac{\partial E_R}{\partial t}} + \vec{\nabla}\vec{F}_R = \int_0^\infty \oint (j_\nu - k_\nu I_\nu)\,d\omega d\nu$$

Total radiated energy

$$E_R(\vec{r},t) = \frac{1}{c}\int_0^\infty \oint I_\nu(\vec{r},\vec{n},\nu,t)\,d\omega d\nu$$

Energy flux

$$\vec{F}_R(\vec{r},t) = \int_0^\infty \oint \vec{n} I_\nu(\vec{r},\vec{n},\nu,t)\,d\omega d\nu$$

Radiation pressure

$$\hat{P}_R(\vec{r},t) = \frac{1}{c}\int_0^\infty \oint \vec{n}\otimes\vec{n} I_\nu(\vec{r},\vec{n},\nu,t)\,d\omega d\nu$$

Two-Fluid description

Average neutral and charged particle

Mass conservation

Neutrals

$$\frac{\partial \rho_n}{\partial t} + \vec{\nabla}(\rho_n \vec{u}_n) = S_n$$

Charges

$$\frac{\partial \rho_c}{\partial t} + \vec{\nabla}(\rho_c \vec{u}_c) = S_i + S_e = S_c$$

$$\rho_c = \sum_\alpha \rho_{\alpha i} + \rho_e$$

....

Two-Fluid description

Average neutral and charged particle

Momentum conservation

Neutrals
$$\rho_n \frac{D\vec{u_n}}{Dt} = \rho_n \vec{g} - \vec{\nabla}\hat{\mathbf{p}}_{\mathbf{n}} + \vec{R}_n - \vec{u}_n S_n$$

Charges
$$\rho_c \frac{D\vec{u}_c}{Dt} = [\vec{J} \times \vec{B}] + \rho_c \vec{g} - \vec{\nabla}\hat{\mathbf{p}}_{\mathbf{ie}} + \vec{R}_{ie} - \vec{u}_c S_c$$

$$\hat{\mathbf{p}}_{\mathbf{n}} = \sum_\alpha \hat{\mathbf{p}}_{\alpha\mathbf{n}} + \sum_\alpha \rho_{\alpha n}(\vec{w}_{\alpha n} \otimes \vec{w}_{\alpha n})$$

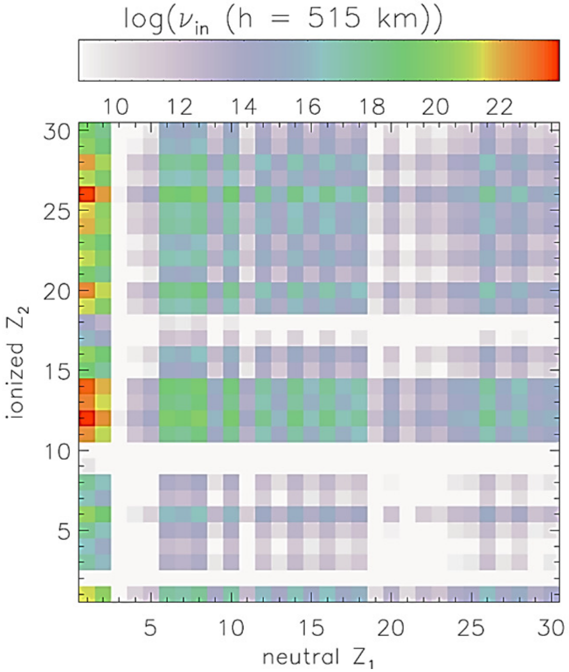
$$\hat{\mathbf{p}}_{\mathbf{ie}} = \sum_\alpha \hat{\mathbf{p}}_{\alpha\mathbf{i}} + \hat{\mathbf{p}}_{\mathbf{e}} + \sum_\alpha \rho_{\alpha i}(\vec{w}_{\alpha i} \otimes \vec{w}_{\alpha i}) + \rho_e(\vec{w}_e \otimes \vec{w}_e)$$

....

Two-Fluid description

Average neutral and charged particle

Momentum conservation



$$\vec{R}_n \approx -\sum_{\beta=1}^{N} \nu_{en_\beta}\rho_e(\vec{u}_n - \vec{u}_e) - \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \nu_{i_\alpha n_\beta}\rho_i(\vec{u}_n - \vec{u}_i)$$

$$\vec{R}_i \approx -\sum_{\alpha=1}^{N} \nu_{ei_\alpha}\frac{m_e}{e}\vec{J} - \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \nu_{i_\alpha n_\beta}\rho_i(\vec{u}_i - \vec{u}_n)$$

$$\vec{R}_e \approx \sum_{\alpha=1}^{N} \nu_{ei_\alpha}\frac{m_e}{e}\vec{J} + \sum_{\beta=1}^{N} \nu_{en_\beta}\rho_e(\vec{u}_n - \vec{u}_e)$$

Two-Fluid description

Average neutral and charged particle

Energy conservation

Neutrals

$$\frac{3}{2}\frac{Dp_n}{Dt} + \frac{3}{2}p_n\vec{\nabla}\vec{u}_n + (\hat{\mathbf{p}}_\mathbf{n}\vec{\nabla})\vec{u}_n + \vec{\nabla}\vec{q}_n =$$

$$\sum_\alpha M_{\alpha n} + \frac{1}{2}u_n^2 S_n - \vec{u}_n\vec{R}_n$$

Charges

$$\frac{3}{2}\frac{Dp_{ie}}{Dt} + \frac{3}{2}p_{ie}\vec{\nabla}\vec{u}_c + (\hat{\mathbf{p}}_\mathbf{ie}\vec{\nabla})\vec{u}_c + \vec{\nabla}\vec{q}_{ie} =$$

$$\vec{J}(\vec{E} + [\vec{u}_c \times \vec{B}]) + \sum_\alpha M_{\alpha i} + M_e + \frac{1}{2}u_c^2 S_c - \vec{u}_c\vec{R}_{ie}$$

Two-Fluid description

Photons

$$\vec{\nabla}\hat{P}_{Rn} = \frac{1}{c}\int_0^\infty \oint (j_\nu - k_\nu I_\nu)\vec{n}\, d\omega\, d\nu \Big|_n$$

$$\vec{\nabla}\hat{P}_{Rc} = \frac{1}{c}\int_0^\infty \oint (j_\nu - k_\nu I_\nu)\vec{n}\, d\omega\, d\nu \Big|_{ie}$$

$$\vec{\nabla}\vec{F}_{Rn} = \int_0^\infty \oint (j_\nu - k_\nu I_\nu)\, d\omega\, d\nu \Big|_n$$

$$\vec{\nabla}\vec{F}_{Rc} = \int_0^\infty \oint (j_\nu - k_\nu I_\nu)\, d\omega\, d\nu \Big|_{ie}$$

Two-Fluid description

Ohm's law

$$\vec{E}^* = [\vec{E} + \vec{u}_c \times B] = \frac{1}{en_e}[\vec{J} \times \vec{B}] - \frac{\vec{\nabla}\hat{\mathbf{p}}_\mathbf{e}}{en_e} + \frac{\rho_e}{(en_e)^2}\left(\sum_\alpha \nu_{ei_\alpha} + \sum_\beta \nu_{en_\beta}\right)\vec{J}$$

$$- \frac{\rho_e}{en_e}(\vec{u}_c - \vec{u}_n)\left(\sum_\beta \nu_{en_\beta} - \sum_\alpha \sum_\beta \nu_{i_\alpha n_\beta}\right)$$

Single-Fluid description

electrons +ions +neutrals

Mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\left(\rho \vec{u}\right) = 0$$

Momentum conservation

$$\rho \frac{D\vec{u}}{Dt} = \vec{J} \times \vec{B} + \rho \vec{g} - \vec{\nabla}\hat{\mathbf{p}} - \vec{\nabla}\hat{\mathbf{P}}_{\mathbf{R}}$$

Energy conservation

$$\frac{\partial}{\partial t}\left(\frac{3p}{2} + \frac{1}{2}\rho u^2\right) + \vec{\nabla}\left(\vec{u}\,(\frac{3p}{2} + \frac{1}{2}\rho u^2) + \hat{\mathbf{p}}\vec{u}\right)$$
$$+ \vec{\nabla}\vec{q} + \vec{\nabla}\vec{F}_R = \vec{J}\vec{E} + \rho\vec{u}\vec{g}$$

Single-Fluid description

$$\vec{E}^* = c_j \vec{J} + c_{jb}[\vec{J} \times \vec{B}] + c_{jbb}[(\vec{J} \times \vec{B}) \times \vec{B}] + c_{pe}\vec{\nabla}\hat{\mathbf{p}}_e$$

$$+ c_{pt}\vec{G} + c_{ptb}[\vec{G} \times \vec{B}]$$

Ohm's law

$$c_j = \frac{1}{(en_e)^2}\left(\sum_{\alpha=1}^{N} \rho_e \nu_{ei_\alpha} + o \sum_{\beta=1}^{N} \rho_e \nu_{en_\beta}\right) \approx \frac{\alpha_e}{(en_e)^2}$$

$$c_{jb} = \frac{1}{en_e}(1 - 2\xi_n \epsilon_1 + \xi_n \epsilon_2)$$

$$c_{jbb} = -\frac{\xi_n^2}{\alpha_n}$$

$$c_{pe} = -\frac{1}{en_e}$$

$$c_{pt} = \frac{1}{en_e}(\epsilon_1 - \epsilon_2)$$

$$c_{ptb} = \frac{\xi_n}{\alpha_n}$$

$$\alpha_n = \sum_{\beta=1}^{N} \rho_e \nu_{en_\beta} + \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \rho_i \nu_{i_\alpha n_\beta}$$

$$\alpha_e = \sum_{\alpha=1}^{N} \rho_e \nu_{ei_\alpha} + \sum_{\beta=1}^{N} \rho_e \nu_{en_\beta}$$

$$\epsilon_1 = \sum_{\beta=1}^{N} \rho_e \nu_{en_\beta}/\alpha_n \ll 1$$

$$\epsilon_2 = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \rho_e \nu_{i_\alpha n_\beta}/\alpha_n \ll 1$$

$$o = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} N\rho_i \nu_{i_\alpha n_\beta}/\alpha_n \approx 1$$