A numerical approach to solving the equations of weakly ionised multifluid MHD

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The Organisations

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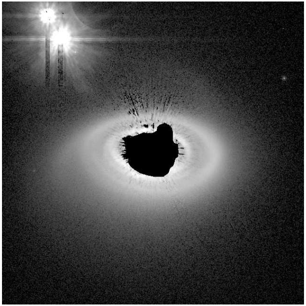
Strong (Hall) diffusion

Protostellar disks



HD141569 Circumstellar Disk NASA HST

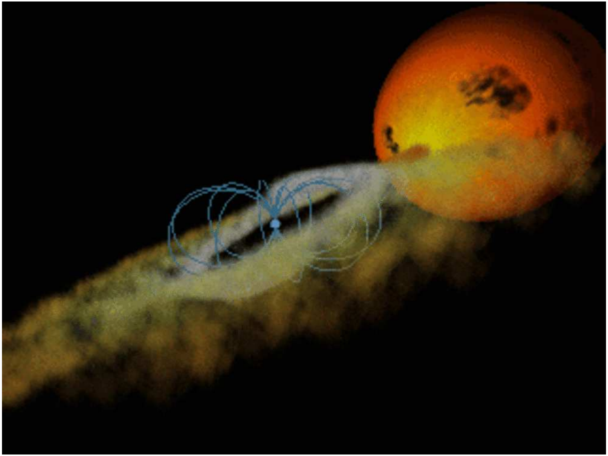
Strong (Hall) diffusion

Dense molecular clouds

Horsehead Nebula NASA HST

Strong (Hall) diffusion

Dwarf nova disk



Derivation of the weakly ionised equations

We assume the equations of MHD, but with a non-zero “diffusive” term in the induction equation - i.e. the electric field in the fluid frame is non-zero.

The equation of motion for the charged species is then

$$\alpha_i \rho_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) = \nabla p_i + \rho_i \frac{D_i}{Dt}\mathbf{v}_i - \sum_{j \neq i}^{N} \mathbf{f}_{ij}$$

One can solve this system numerically, but better to simplify it if possible.

Weakly ionised approximation

- We presume that a neutral species dominates (by mass) the system and thus the overall velocity of the fluid(s) is roughly $\mathbf{v}_1$.
- Neglect collisions between different charged species
- Neglect inertia (and pressure) of charged species

Weakly ionised approximation

The equation of motion of the charged species becomes

$$\alpha_i \rho_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) + \mathbf{f}_{i1} = 0$$

where

$$\mathbf{f}_{ij} = \mathbf{C}_{ij} + s_{ij} \mathbf{v}_j - s_{ji} \mathbf{v}_i$$

where

$$\mathbf{C}_{ij} = \rho_i \rho_j K_{ij} (\mathbf{v}_j - \mathbf{v}_i)$$

(we neglect terms with s, and K is related to the collision frequency)

Weakly ionised approximation

Now

$$\mathbf{J} = \sum_{i=1}^{N} \alpha_i \rho_i \mathbf{v}_i$$

We assume

$$\sum_{i=1}^{N} \alpha_i \rho_i = 0$$

and re-write our equations in the rest-frame of the neutral fluid:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$
$$0 = \alpha_i \rho_i (\mathbf{E}' + \mathbf{v}_i' \times \mathbf{B}) - \frac{B}{\beta_i} (\alpha_i \rho_i \mathbf{v}_i').$$

where we introduce the Hall parameter, $\beta_i \equiv \frac{\alpha_i B}{K_{1i} \rho_1}$.

Weakly ionised approximation

Calculating **J** from our equations of motion for the charged species we arrive (after a little algebra) at

$$\mathbf{J} = \sigma_{||}\mathbf{E}'_{||} + \sigma_{\perp}\mathbf{E}' + \sigma_{\mathrm{H}}(\mathbf{E}' \times \mathbf{b})$$

where

$$\sigma_{\mathrm{H}} = \frac{1}{B}\sum_{i=1}^{N} \frac{\alpha_i \rho_i}{1+\beta_i^2},$$

$$\sigma_{\perp} = \frac{1}{B}\sum_{i=1}^{N} \frac{\alpha_i \rho_i \beta_i}{1+\beta_i^2},$$

$$\sigma_{||} = \frac{1}{B}\sum_{i=1}^{N} \alpha_i \rho_i \beta_i .$$

Weakly ionised approximation

We can invert this equation to get

$$\mathbf{E}' = r_0 \frac{(\mathbf{J} \cdot \mathbf{B})\mathbf{B}}{B^2} + r_1 \frac{\mathbf{B} \times \mathbf{J}}{B} + r_2 \frac{\mathbf{B} \times (\mathbf{J} \times \mathbf{B})}{B^2}.$$

This is our Generalised Ohm's Law for the weakly ionised case.

The weakly ionised equations

Our equations for our system are then:

$$
\begin{aligned}
\frac{\partial \rho_i}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_i \boldsymbol{v}_i) &= 0, (1 \le i \le N), \\
\frac{\partial \rho_1 \boldsymbol{v}_1}{\partial t} + \nabla \cdot \left(\rho \boldsymbol{v}_1 \boldsymbol{v}_1 + a^2 \rho \mathbf{I} \right) &= \boldsymbol{J} \times \boldsymbol{B}, \\
\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}_1 \boldsymbol{B} - \boldsymbol{B} \boldsymbol{v}_1) &= -\nabla \times \boldsymbol{E}', \\
\alpha_i \rho_i (\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}) + \rho_i \rho_1 K_{i\,1} (\boldsymbol{v}_1 - \boldsymbol{v}_i) &= 0, 2 < i \le N, \\
\nabla \cdot \boldsymbol{B} &= 0, \\
\nabla \times \boldsymbol{B} &= \boldsymbol{J}, \\
\sum_{i=2}^{N} \alpha_i \rho_i &= 0.
\end{aligned}
$$

Numerical technique

If we know $\boldsymbol{B}$ we can advance these equations easily:

- Advance the neutral density and neutral momentum equations using an explicit upwind method (without source terms)
- Advance the charged density equations
- Apply the source term to the neutral momentum equation
- Advance magnetic field
- Algebraically solve for the charged gas velocities

This is done in an operator-split fashion

The induction equation

Consider the linearised induction equation

$$\frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{M}}{\partial x} = r_{\mathrm{O}} \mathbf{R}_{\mathrm{O}} \frac{\partial^2 \mathbf{B}}{\partial x^2} + r_{\mathrm{H}} \mathbf{R}_{\mathrm{H}} \frac{\partial^2 \mathbf{B}}{\partial x^2} + r_{\mathrm{A}} \mathbf{R}_{\mathrm{A}} \frac{\partial^2 \mathbf{B}}{\partial x^2}$$

where

$$\mathbf{B} = (B_y, B_z)$$
$$\mathbf{M} = (uB_y - vB_x, uB_z - wB_x)$$

The challenge

Discretise the induction equation, ignoring $r_{\rm O}$ and assuming $\mathbf{B}_j^n = \mathbf{B}^n e^{i\omega j}$ to get

$$\mathbf{B}^{n+1} = \mathbf{A}\mathbf{B}^n$$

Stability requires the spectral radius of **A** to be less than or equal to 1

Conventional methods

- Explicit methods:
  - Conventional schemes show poor stability for large Hall effect ($\Delta t \to 0$ as $\eta \equiv \frac{r_{\mathrm{A}}}{r_{\mathrm{H}}}$ becomes small)
  - Even for low η they require quite a low time-step ($\Delta t \propto (\Delta x)^{-2}$)
  - Simple to implement in multidimensional/parallel/adaptive codes
- Implicit methods:
  - Unconditionally stable
  - Difficult to implement in multidimensional/parallel/adaptive codes

We want it all: An explicit, stable method

Dealing with $\nabla \times \mathbf{E}'$

For diagonal terms (associated with r_2 and r_0 if large) use Super Time-Stepping:

- $\tau^{\mathrm{STS}} = \sum_{i=1}^{N} d\tau_i$
- Require that $\left|\prod_{i-1}^{N}(1 - d\tau_i \hat{\lambda})\right| < 1$
- $d\tau_i = \tau^{\mathrm{STD}} \left[(-1+\nu)\cos\left(\frac{2i-1}{N}\frac{\pi}{2}\right) + 1 + \nu\right]^{-1}$
- N and ν are user-defined
- $\tau^{\mathrm{STS}} \to N^2 \tau^{\mathrm{STD}}$ as $\nu \to 0$

Dealing with $\nabla \times \mathbf{E}'$

For the Hall term (r_1 term) use the “Hall Diffusion Scheme” which relies on the off-diagonal nature of the Hall operator.
An example: For a pure Hall operator (anti-symmetric operator) we might have the following discretisation:

$$(B_y)_i^{n+1} = (B_y)_i^n + \frac{\Delta t}{\Delta x^2}\left[(B_z)_{i+1}^n - 2(B_z)_i^n + (B_z)_{i-1}^n\right]$$
$$(B_z)_i^{n+1} = (B_z)_i^n + \frac{\Delta t}{\Delta x^2}\left[(B_y)_{i+1}^{n+1} - 2(B_y)_i^{n+1} + (B_y)_{i-1}^{n+1}\right]$$

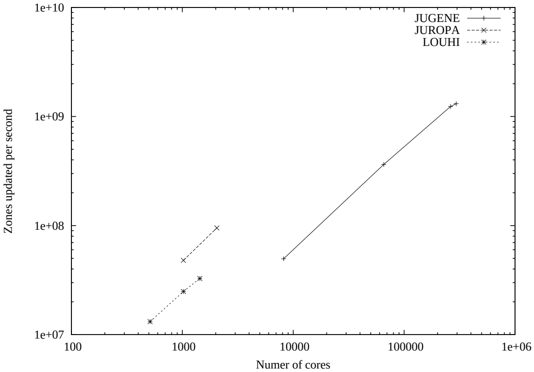
Explicit for B_y, implicit for B_z. Stability is good, though still have $\Delta t \propto \Delta x^2$.

How efficient is the resulting code?

For some standard tests we get the following timings (in seconds, for a serial code):

	Ambipolar dominated	Hall dominated	Sub-shock
Implicit	1.9	23.3	2.7
Explicit	1.9	14.8	1.9

Do the algorithms scale?



- Scaling for JuRoPA system is virtually perfect up to 2048 cores.
- Code scales (on BG/P systems) from 8192 cores up to 294912 cores with circa 70% efficiency.
- Performance on XT5 has since been improved ...

The importance of scaling

For some published turbulence simulations:

- On an 8-core workstation a 64^3 run takes about 12 hrs
- On the BG/Q with 4096 cores a 512^3 run takes about 1 day
- On an 8-core workstation a 512^3 run would take at least 6 years

Conclusions

- The weakly ionised approximation is very useful (approach can be extended to fully ionised)
- Explicit algorithms are both available and valuable
- Algorithm development should, if possible, allow for scaling
- There are still outstanding issues ....
  - What can be done if diffusion is very large?
  - Diffusion in rarefied regions sometimes dominates - can we ignore it?