Alfvén Waves in Partially Ionized Plasmas: Frequency Cut-offs and the Role of Hall's Term

Roberto Soler[1]

J.L. Ballester[1], M. Carbonell[1], J. Terradas[1], T. V. Zaqarashvili[2]

[1]Solar Physics Group, Universitat de les Illes Balears (Spain)
[2]Space Research Institute, Austrian Academy of Sciences (Austria)

International Team on Partially Ionized Plasmas in Astrophysics
Bern, 27–31 January 2014

Outline

Motivation: Are There Cut-offs for Alfvén Waves in PIP?

- Cut-off wavenumbers for Alfvén waves have been found in the single-fluid approximation
  e.g., Balsara (1996), Soler et al. (2009), Barceló et al. (2011)
- The physical existence of these cut-offs has been disputed: it is argued that they are a *"mathematical artefact not connected to any real physical process"* (Zaqarashvili et al. 2011, 2012)

- Cut-off wavenumbers appear in the (more general) two-fluid theory
  Kulsrud & Pierce (1969), Pudritz (1990), Soler et al. (2013)
- The two-fluid cut-offs have been explained in physical terms
  Mouschovias (1987), Kamaya & Nishi (1998), Soler et al. (2013)

- How do we reconcile single-fluid and two-fluid results?
- What is the role of Hall's current? (Zaqarashvili et al. 2012)
- What are the implications for Alfvén waves in the chromosphere?

Motivation: Are There Cut-offs for Alfvén Waves in PIP?

- Cut-off wavenumbers for Alfvén waves have been found in the single-fluid approximation
  e.g., Balsara (1996), Soler et al. (2009), Barceló et al. (2011)
- The physical existence of these cut-offs has been disputed: it is argued that they are a *"mathematical artefact not connected to any real physical process"* (Zaqarashvili et al. 2011, 2012)

- Cut-off wavenumbers appear in the (more general) two-fluid theory
  Kulsrud & Pierce (1969), Pudritz (1990), Soler et al. (2013)
- The two-fluid cut-offs have been explained in physical terms
  Mouschovias (1987), Kamaya & Nishi (1998), Soler et al. (2013)

- How do we reconcile single-fluid and two-fluid results?
- What is the role of Hall's current? (Zaqarashvili et al. 2012)
- What are the implications for Alfvén waves in the chromosphere?

Motivation: Are There Cut-offs for Alfvén Waves in PIP?

- Cut-off wavenumbers for Alfvén waves have been found in the single-fluid approximation
  e.g., Balsara (1996), Soler et al. (2009), Barceló et al. (2011)
- The physical existence of these cut-offs has been disputed: it is argued that they are a *"mathematical artefact not connected to any real physical process"* (Zaqarashvili et al. 2011, 2012)

- Cut-off wavenumbers appear in the (more general) two-fluid theory
  Kulsrud & Pierce (1969), Pudritz (1990), Soler et al. (2013)
- The two-fluid cut-offs have been explained in physical terms
  Mouschovias (1987), Kamaya & Nishi (1998), Soler et al. (2013)

- How do we reconcile single-fluid and two-fluid results?
- What is the role of Hall's current? (Zaqarashvili et al. 2012)
- What are the implications for Alfvén waves in the chromosphere?

Single-fluid Approximation: Basic Equations

- Ions, electrons, and neutrals considered as a single fluid
- Electron inertia neglected
- Ion-neutral collisions remain through ambipolar diffusion term
- Electron-neutral collisions neglected here!

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{b}) \times \mathbf{B}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta_{\mathrm{A}} \frac{1}{B^2} \nabla \times \{[(\nabla \times \mathbf{b}) \times \mathbf{B}] \times \mathbf{B}\}$$
$$-\eta \nabla \times \nabla \times \mathbf{b} - \eta_{\mathrm{H}} \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{B}]$$

$$\rho = \rho_{\mathrm{i}} + \rho_{\mathrm{n}}, \qquad p = p_{\mathrm{i}} + p_{\mathrm{e}} + p_{\mathrm{n}}, \qquad \mathbf{v} = \frac{\rho_{\mathrm{i}} \mathbf{v}_{\mathrm{i}} + \rho_{\mathrm{n}} \mathbf{v}_{\mathrm{n}}}{\rho_{\mathrm{i}} + \rho_{\mathrm{n}}},$$

$$\eta = \frac{\alpha_{\mathrm{ie}}}{\mu e^2 n_e^2}, \qquad \eta_{\mathrm{A}} = \left(\frac{\rho_{\mathrm{n}}}{\rho_{\mathrm{i}} + \rho_{\mathrm{n}}}\right)^2 \frac{B^2}{\mu \alpha_{\mathrm{in}}}, \qquad \eta_{\mathrm{H}} = \frac{1}{\mu e n_e}$$

Alfvén Wave Dispersion Relation (Without Hall's Term)

- Hall's term is dropped
- Fourier analysis of perturbations: $\exp\left(ikz - i\omega t\right)$
- Dispersion relation of incompressible ($\nabla \cdot \mathbf{v} = 0$) Alfvén waves

$$\omega^2 + ik^2\eta_{\mathrm{C}}\omega - \frac{k^2 v_{\mathrm{A,i}}^2}{1+\chi} = 0$$

$$\chi = \frac{\rho_{\mathrm{n}}}{\rho_{\mathrm{i}}}, \qquad v_{\mathrm{A,i}} = \frac{B}{\sqrt{\mu\rho_{\mathrm{i}}}}, \qquad \eta_{\mathrm{C}} = \eta_{\mathrm{A}} + \eta$$

- Solution to the dispersion relation

$$\omega = \pm \frac{k v_{\mathrm{A,i}}}{\sqrt{1+\chi}} \left[1 - \frac{k^2\eta_{\mathrm{C}}^2}{4 v_{\mathrm{A,i}}^2}\left(1+\chi\right)\right]^{1/2} - i\frac{k^2\eta_{\mathrm{C}}^2}{2}$$

Single-fluid Cut-off Wavenumber (Without Hall's Term)

- Frequency cut-off $\rightarrow \mathrm{Re}(\omega) = 0$

$$k_{\mathrm{SF}} = \frac{2 v_{\mathrm{A,i}}}{\sqrt{1+\chi}} \frac{1}{\eta_{\mathrm{C}}}$$

- If $k < k_{\mathrm{SF}} \rightarrow$ Oscillatory modes (complex frequency)
- If $k > k_{\mathrm{SF}} \rightarrow$ Evanescent modes (purely imaginary frequency)

- Weakly ionized plasmas, $\chi \gg 1$

$$\eta_{\mathrm{C}} = \eta_{\mathrm{A}} + \eta \overset{\chi \gg 1}{\approx} \eta_{\mathrm{A}} = \frac{v_{\mathrm{A,i}}^2}{\nu_{\mathrm{ni}}} \frac{\chi}{(1+\chi)^2}, \qquad \nu_{\mathrm{ni}} = \frac{\alpha_{\mathrm{in}}}{\rho_{\mathrm{n}}}$$

$$k_{\mathrm{SF}} \approx \frac{\nu_{\mathrm{ni}}}{v_{\mathrm{A,i}}} \frac{2(1+\chi)^{3/2}}{\chi} \overset{\chi \gg 1}{\approx} \frac{2\sqrt{\chi}\,\nu_{\mathrm{ni}}}{v_{\mathrm{A,i}}}$$

- **In the single-fluid approximation there is only one critical k**
- There is no threshold χ. Single-fluid cut-off present for any χ

Single-fluid Cut-off Wavenumber (Without Hall's Term)

Physical/Mathematical Interpretation

- Order-of-magnitude analysis

$$\frac{[\text{Ambipolar term}]}{[\text{Inductive term}]} \sim \frac{\eta_{\rm A} B/L^2}{v_{\rm A,i} B/\sqrt{1+\chi} L} \sim \frac{v_{\rm A,i}}{\nu_{\rm ni}} \frac{\chi}{(1+\chi)^{3/2}} \frac{1}{L}$$

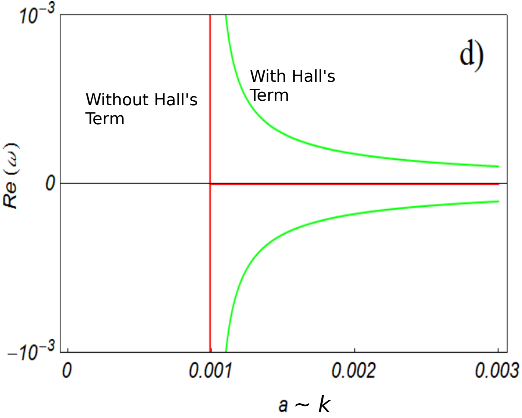
$$\frac{[\text{Ambipolar term}]}{[\text{Inductive term}]} \sim 1 \qquad \text{if} \qquad L^{-1} \sim \frac{\nu_{\rm ni}}{v_{\rm A,i}} \frac{(1+\chi)^{3/2}}{\chi} = \frac{1}{2} k_{\rm SF}$$

- $k_{\rm SF}$ defines a length scale at which the ambipolar term becomes of the same importance as the inductive term
- For $k > k_{\rm SF}$ magnetic field perturbations are dominated by ambipolar diffusion: wave propagation is suppressed

Single-fluid Result With Hall's Term

Zaqarashvili et al. (2012)

- The role of Hall's term was studied by Zaqarashvili et al. (2012)
- The strict cut-off due to Cowling's diffusion is removed
- Instead, $\mathrm{Re}(\omega)$ takes very small values for $k > k_{\mathrm{SF}}$



Two-fluid Theory: Basic Equations

- Ion-electrons and neutrals are separate fluids
- Interaction by means of ion-neutral collisions

$$\rho_{\rm i}\frac{\partial \mathbf{v}_{\rm i}}{\partial t} = -\nabla\left(p_{\rm i}+p_{\rm e}\right)+\frac{1}{\mu}\left(\nabla\times\mathbf{b}\right)\times\mathbf{B}-\alpha_{\rm in}\left(\mathbf{v}_{\rm i}-\mathbf{v}_{\rm n}\right)$$

$$\rho_{\rm n}\frac{\partial \mathbf{v}_{\rm n}}{\partial t} = -\nabla p_{\rm n}-\alpha_{\rm in}\left(\mathbf{v}_{\rm n}-\mathbf{v}_{\rm i}\right)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla\times\left(\mathbf{v}_{\rm i}\times\mathbf{B}\right)-\eta\nabla\times\nabla\times\mathbf{b}-\eta_{\rm H}\nabla\times\left[\left(\nabla\times\mathbf{b}\right)\times\mathbf{B}\right]$$

- Fourier analysis of perturbations: $\exp\left(ikz-i\omega t\right)$
- Dispersion relation of Alfvén waves without Hall's term

$$\omega^3 + i\left[k^2\eta+\left(1+\chi\right)\nu_{\rm ni}\right]\omega^2 - \left[k^2 v_{\rm A,i}^2+k^2\eta\left(1+\chi\right)\nu_{\rm ni}\right]\omega - i\nu_{\rm ni}k^2 v_{\rm A,i}^2 = 0$$

Two-fluid Cut-off Wavenumbers (Without Hall's Term)

- The dispersion relation is a third-order polynomial
- The presence of cut-offs is discussed using the discriminant
- Three different cut-off wavenumbers are found: k_1, k_2, and k_3

- Cut-off wavenumbers k_1 and k_2 only possible when $\chi > 8$

$$k_{1,2} \approx \frac{\nu_{\rm ni}}{v_{\rm A,i}} \left[\frac{\chi^2 + 20\chi - 8}{8\left(1+\chi\right)^3} \pm \frac{\chi^{1/2}\left(\chi-8\right)^{3/2}}{8\left(1+\chi\right)^3} \right]^{-1/2} \xrightarrow{\chi \gg 1} \begin{cases} k_1 \approx \frac{2\sqrt{\chi}\nu_{\rm ni}}{v_{\rm A,i}} \\ k_2 \approx \frac{\chi \nu_{\rm ni}}{2 v_{\rm A,i}} \end{cases}$$

- If $\chi < 8 \longrightarrow$ Oscillatory modes
- If $\chi > 8$:
  - For $k \notin [k_1, k_2] \longrightarrow$ Oscillatory modes
  - For $k \in [k_1, k_2] \longrightarrow$ Evanescent modes $\longrightarrow$ **Cut-off interval due to ion-neutral collisions!** (Kulsrud & Pierce 1969)

- Cut-off wavenumber k_3 caused by Ohmic diffusion: $k_3 \approx 2 v_{\rm A,i}/\eta$
- If $k > k_3 \longrightarrow$ Evanescent modes

Two-fluid Cut-off Wavenumbers (Without Hall's Term)

Physical Interpretation

- Order-of-magnitude analysis (neutrals move with ions)

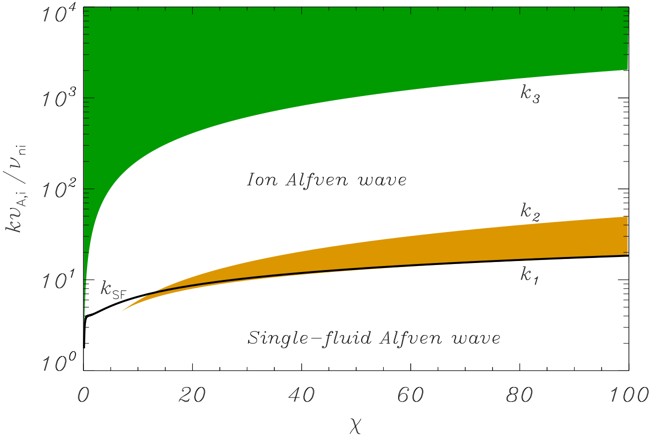
$$\frac{[\text{Friction force}]}{[\text{Magnetic force}]} \sim \frac{\alpha_{\rm in} v_{\rm A,i}/\sqrt{\chi}}{B^2/\mu L} \sim \frac{\sqrt{\chi}\nu_{\rm ni}}{v_{\rm A,i}} L \sim 1 \quad \text{if} \quad L^{-1} \sim \frac{\sqrt{\chi}\nu_{\rm ni}}{v_{\rm A,i}} \sim 2k_1$$

- Order-of-magnitude analysis (static neutrals)

$$\frac{[\text{Friction force}]}{[\text{Magnetic force}]} \sim \frac{\alpha_{\rm in} v_{\rm A,i}}{B^2/\mu L} \sim \frac{\chi\nu_{\rm ni}}{v_{\rm A,i}} L \sim 1 \quad \text{if} \quad L^{-1} \sim \frac{\chi\nu_{\rm ni}}{v_{\rm A,i}} \sim \frac{1}{2}k_2$$

- If $k < k_1 \rightarrow$ Ions and neutrals move as a single fluid
- If $k_1 < k < k_2 \rightarrow$ Friction force $>$ Magnetic force $\rightarrow$ No oscillations
- If $k_2 < k < k_3 \rightarrow$ Ions move, neutrals static
- If $k > k_3 \rightarrow$ Ohmic diffusion dominates $\rightarrow$ No oscillations

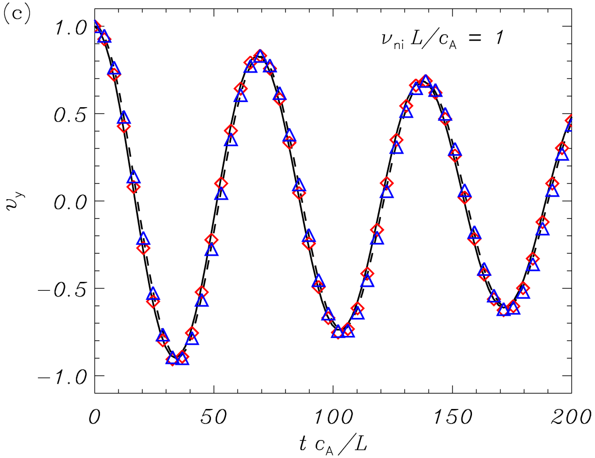
Two-fluid vs. Single-fluid (Without Hall's Term)



- The single-fluid cut-off is (an approximation to) a physical cut-off
- When $\chi > 8$, $k_{\rm SF} \approx k_1$ (but single-fluid ignores k_2 and k_3)
- When $\chi < 8$, $k_{\rm SF}$ underestimates k_3.

Numerical Experiment (Without Hall's Term)

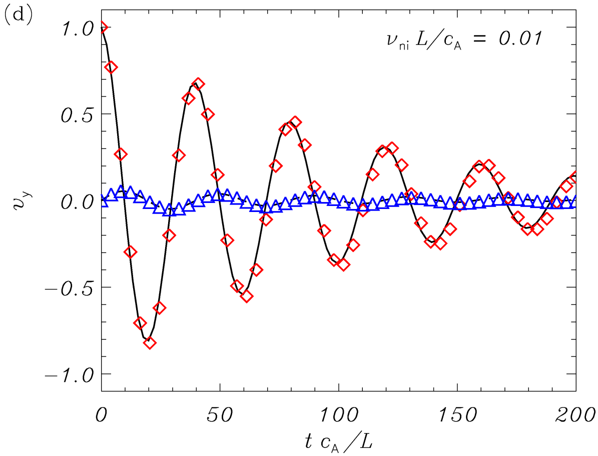
1D simulation of a standing wave with $k < k_1$



◇ = Ions △ = Neutrals

Numerical Experiment (Without Hall's Term)

1D simulation of a standing wave with $k_2 < k < k_3$



◇ = Ions △ = Neutrals

Two-fluid Results With Hall's Term

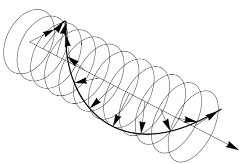
- Dispersion relation of Alfvén waves with Hall's term

$$\Big(\omega^3 + i\left[k^2\eta + (1+\chi)\,\nu_{\mathrm{ni}}\right]\omega^2 - \left[k^2 v_{\mathrm{A,i}}^2 + k^2\eta\,(1+\chi)\,\nu_{\mathrm{ni}}\right]\omega - i\nu_{\mathrm{ni}}k^2 v_{\mathrm{A,i}}^2\Big)^2 - k^4 B^2 \eta_{\mathrm{H}}^2 \omega^2 \left[\omega + i\,(1+\chi)\,\nu_{\mathrm{ni}}\right]^2 = 0$$

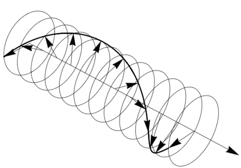
- Due Hall's term the number of solution gets doubled!
- Reason: left and right circularly polarized waves have different frequencies when Hall's current is at work

Zhelyazkov et al. (1996); Cramer (2001); Pécseli (2013)

Left circularly polarized electric field



Right circularly polarized electric field



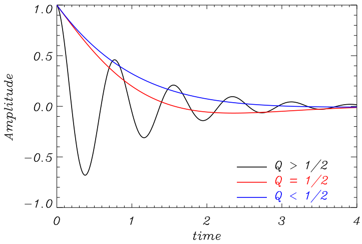
Effective Cut-offs and the Quality Factor

- Strict cut-offs are removed due to Hall's term as in the single-fluid case (Zaqarashvili et al. 2012)
- What is the practical implication of this result?

- The quality factor compares the frequency at which a wave oscillates to the rate at which it damps

$$Q \equiv \frac{1}{2}\left|\frac{\mathrm{Re}(\omega)}{\mathrm{Im}(\omega)}\right|$$

- $Q \rightarrow \infty \rightarrow$ Undamped
- $1/2 < Q < \infty \rightarrow$ Underdamped
- $Q = 1/2 \rightarrow$ Critically damped
- $0 < Q < 1/2 \rightarrow$ Overdamped
- $Q = 0 \rightarrow$ Evanescent (cut-off)



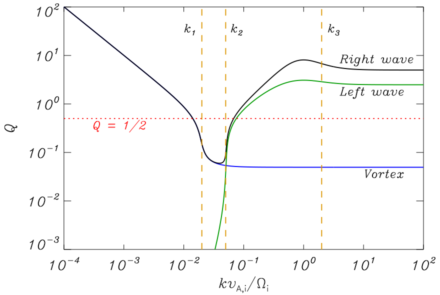
- Nonoscillatory behavior if $Q < 1/2 \rightarrow$ **Effective cut-off!**

Effect of Hall's Term on Q

- Order-of-magnitude analysis

$$\frac{[\text{Hall's term}]}{[\text{Inductive term}]} \sim \frac{\eta_{\rm H} B^2/L^2}{v_{\rm A,i} B/L} \sim \frac{k v_{\rm A,i}}{\Omega_{\rm i}}$$

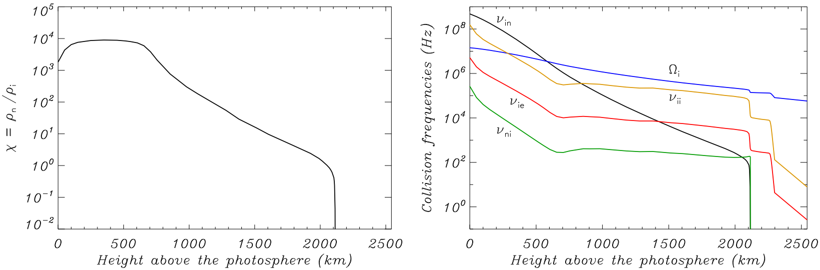
$$L^{-1} \sim k, \qquad \Omega_{\rm i} = \frac{eB}{m_{\rm i}}$$



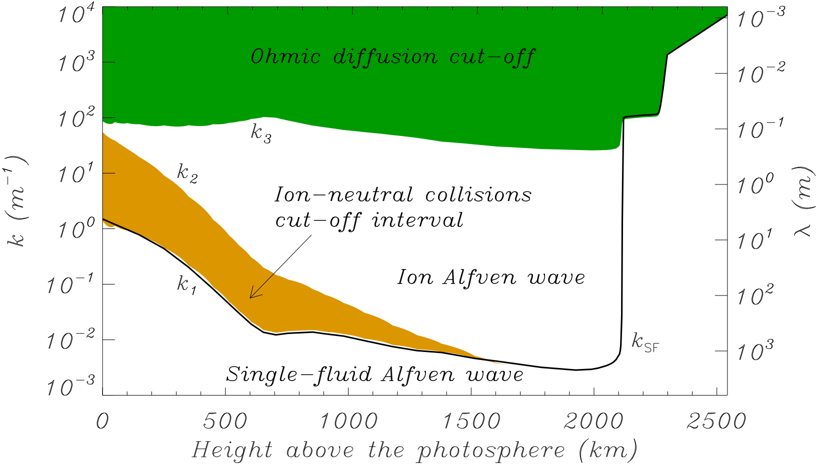
- Ion-neutral cut-off replaced by overdamping ($Q < 1/2$)
- Ohmic cut-off completely removed ($Q > 1/2$)

Application to the Chromosphere

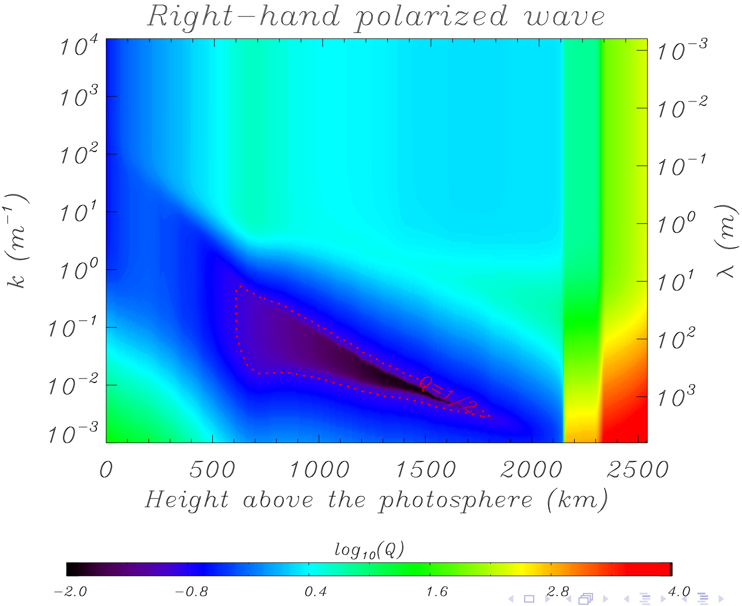
- Dependence with height of physical parameters from VALC model
  Vernazza et al. (1981)
- Only hydrogen is taken into account (no helium!)
- Magnetic field strength: $B = B_{\mathrm{ph}} \left(\frac{\rho}{\rho_{\mathrm{ph}}} \right)^{0.3}$, $B_{\mathrm{ph}} = 1.5$ kG
  Leake & Arber (2006)
- Ion-neutral collision cross section $\sigma_{\mathrm{in}} = 5 \times 10^{-19}$ m^2



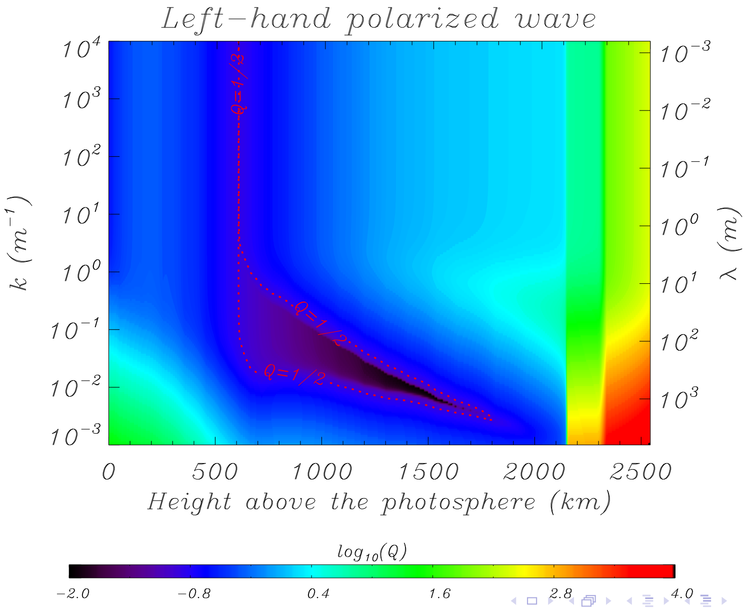
Cut-off Wavenumbers without Hall's Term



Result with Hall's Term



Result with Hall's Term



Conclusions

General results

- Left and right circularly polarized waves have different frequencies (and different Q) when Hall's term is included
- Strict frequency cut-offs of Alfvén waves in partially ionized plasmas are absent due to Hall's term ($Q \neq 0$)

Chromospheric waves

- The cut-off interval due to ion-neutral collisions is replaced by a smaller region where the waves are overdamped ($Q < 1/2$)
- The cut-off due to Ohmic diffusion is completely removed and the waves become underdamped ($Q > 1/2$)

- Hall's term should be taken into account in the studies of wave dynamics in partially ionized plasmas

Future Improvements and Open Questions

- To take helium into account
- To include viscosity
- Role of ionization and recombination
- Validity of the two-fluid theory in the low chromosphere
- Is a three-fluid theory necessary?
- Is a hybrid theory (fluid + kinetic) necessary?
- ...

Thank you for your attention!

Future Improvements and Open Questions

- To take helium into account
- To include viscosity
- Role of ionization and recombination
- Validity of the two-fluid theory in the low chromosphere
- Is a three-fluid theory necessary?
- Is a hybrid theory (fluid + kinetic) necessary?
- ...

Thank you for your attention!