Connecting the Micro-dynamics to the Emergent Macro-variables: Self-Organized Criticality and Absorbing Phase Transitions in the Deterministic Lattice Gas

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Outline

1. From micro to macro: coarse graining
2. Temporal fluctuations: an example
3. The relevant effective Eq of Motion
4. Size dependence
5. High and low density
6. Nearby phase transition
7. Conclusion

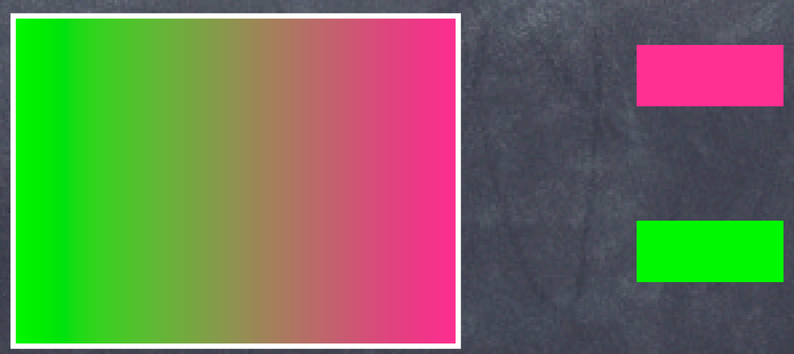
Micro level

$$\frac{dx_i}{dt} = F_i(x_1, x_2, ..., x_N, \dot{x}_1, \dot{x}_2, ..\dot{x}_N, ...) \ for \ i = 1, ..., N$$

Trajectories

Macro level

$$\frac{dn(\mathbf{x})}{dt} = F[n, \nabla n, ..]$$



High density

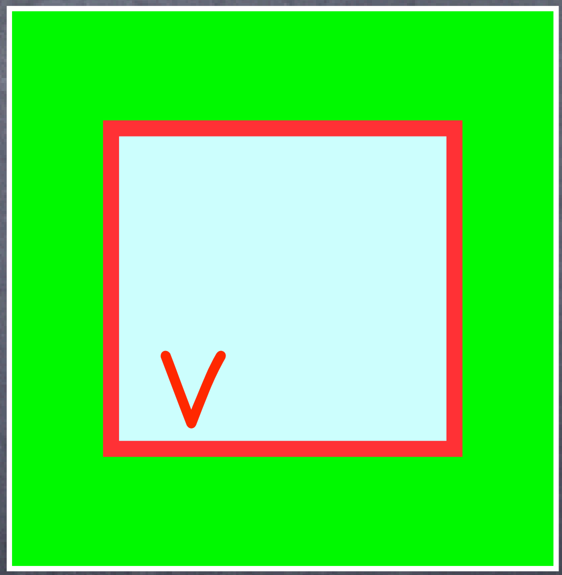
Low density

Charaterising the macro dynamics

Correlations $C(r,t) = \langle n(0,0)n(\mathbf{x},t)\rangle - \langle n(0,0)\rangle^2$

Study sum-volume $N(t) = \int_V d\mathbf{x} n(\mathbf{x},t)$

$$C(t) = \langle N(0)N(t)\rangle - \langle N(0)\rangle^2$$



and power spectra

$$S_N(\omega) = |\hat{N}(\omega)|^2 = \int_{-\infty}^{\infty} dt C(t) e^{i\omega t}$$

$$S_N(\omega) = |\hat{N}(\omega)|^2 = \int_{-\infty}^{\infty} dt C(t) e^{i\omega t}$$

Assume scaling, i.e. power law, behaviour $C(t) \propto 1/t^\kappa$

$$S_N(\omega) \propto \int_{-\infty}^{\infty} dt \frac{1}{t^\kappa} e^{i\omega t} \quad = \omega^{-1+\kappa} \int_{-\infty}^{\infty} du u^{-\kappa} e^{iu}$$

Which suggests

$$C(t) \sim 1/t^\kappa$$

$$S_N(\omega) \sim 1/\omega^\mu$$

$$\Rightarrow \quad \kappa + \mu = 1$$

Hence the interest in $\mu \approx 1$

Which were at the heart of the motivation for Self-Organised Criticality

PHYSICAL REVIEW LETTERS

VOLUME 59 | 27 JULY 1987 | NUMBER 4

Self-Organized Criticality: An Explanation of $1/f$ Noise

Per Bak, Chao Tang, and Kurt Wiesenfeld

Physics Department, Brookhaven National Laboratory, Upton, New York 11973

(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or $1/f$ noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

PACS numbers: 05.40.+j, 02.90.+p

But the sandpile turned out not to possess 1/f (nor fractals)

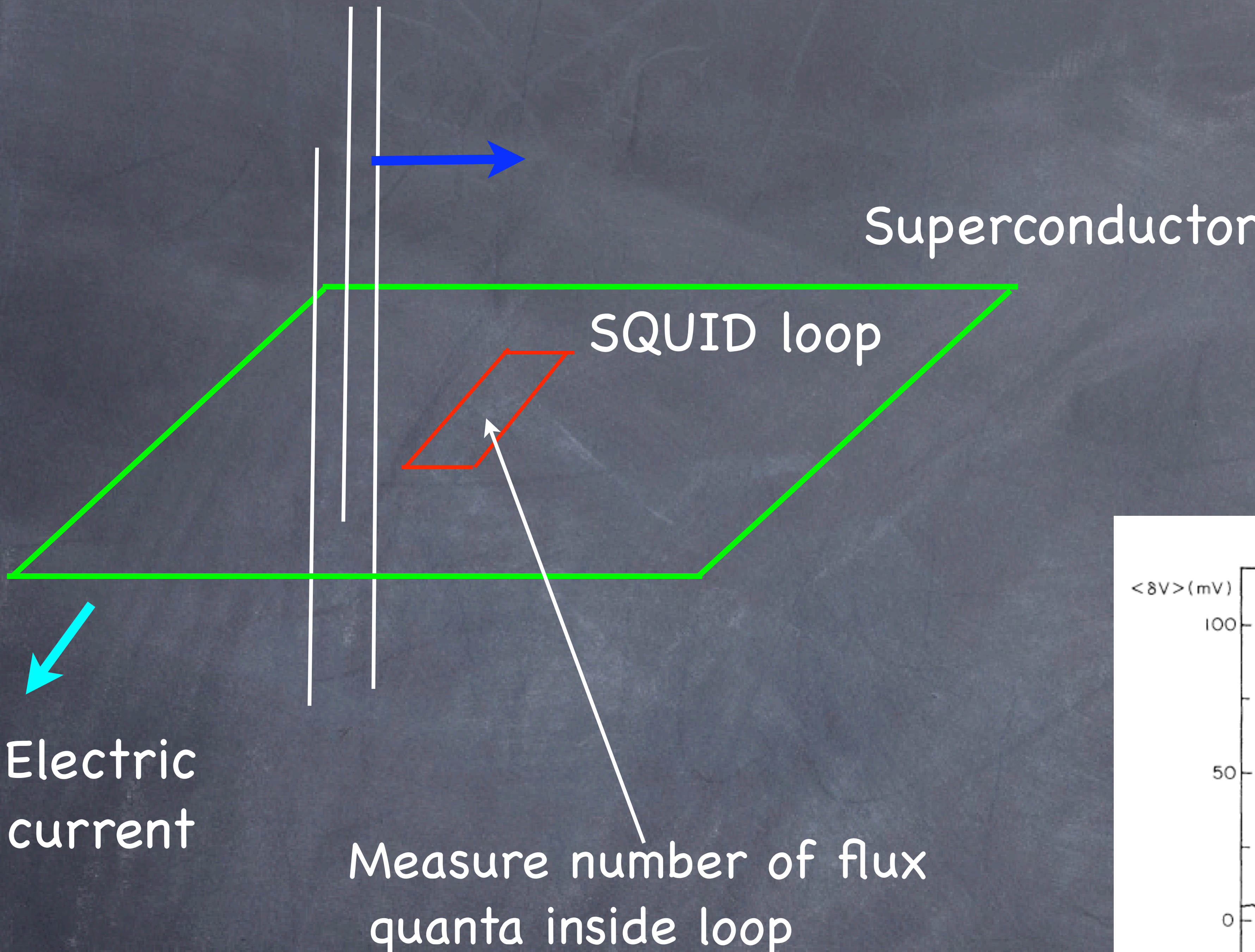
Granular piles "problematic"
so which experiment is then paradigmatic

VOLUME 53, NUMBER 16 PHYSICAL REVIEW LETTERS 15 OCTOBER 1984

Measurements of Flux-Flow and $1/f$ Noise in Superconductors

W. J. Yeh and Y. H. Kao
Department of Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 11 July 1984)

Direct measurements of fluctuations associated with flux flow in superconductors are made with a superconducting quantum interference device. This method avoids the complicated problems concerned with voltage probe locations in earlier work, and allows an unambiguous determination of some characteristic noise power spectra in different current regimes. Models of fluxoid motion are proposed to account for the experimental results.



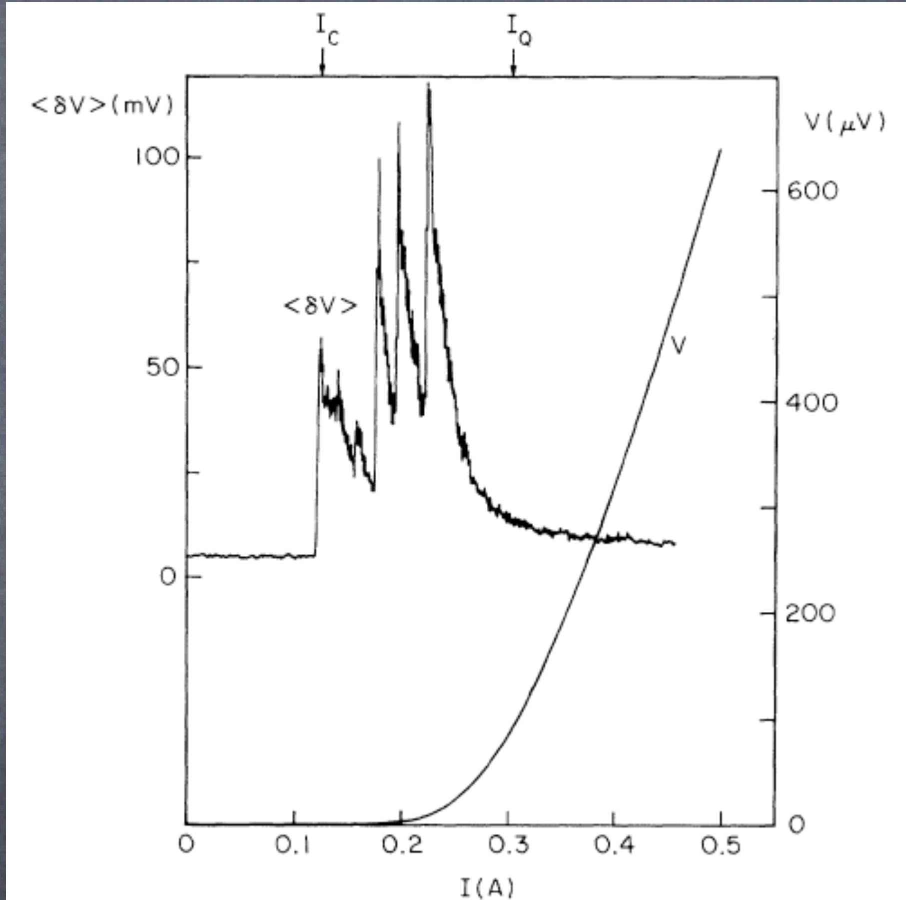
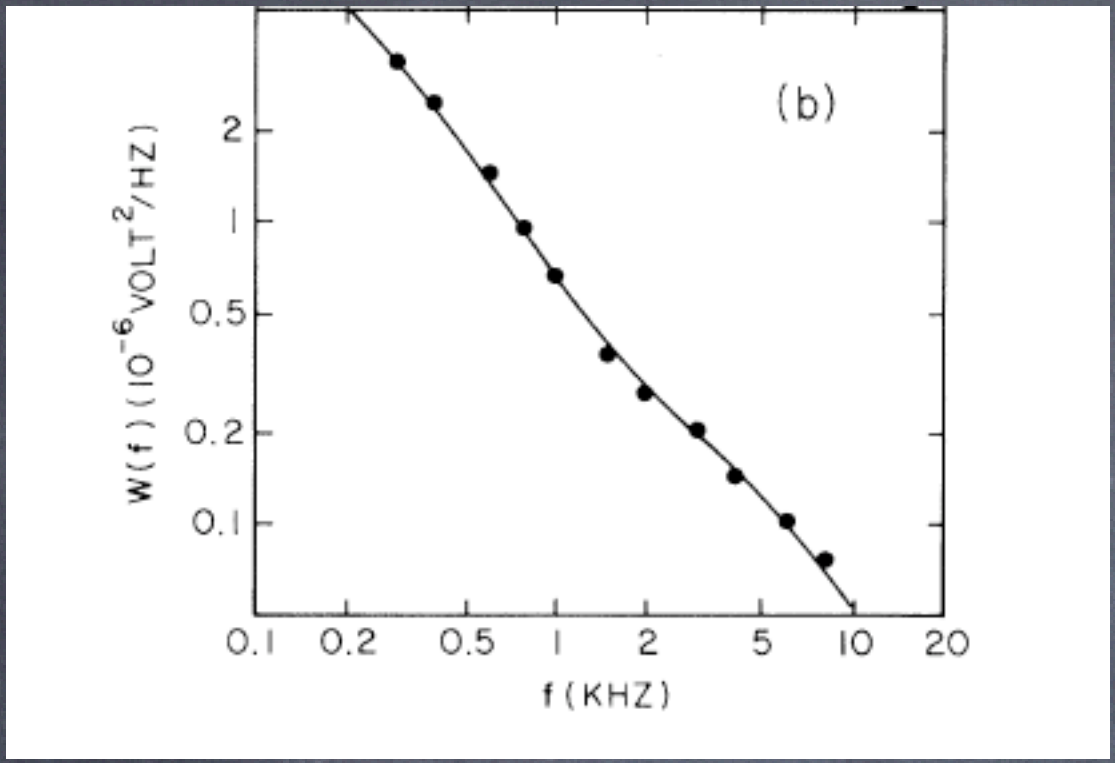


FIG. 1. A representative plot of $\langle \delta V \rangle$ vs transport current I, obtained with $H = 227.5$ Oe and $T = 3.672$ K. I_c marks the critical current at the onset of flux flow. I_Q separates the near-onset region from the quasilinear region. A V-I curve is included for comparison.

Number of flux lines in loop N(t) fluctuates in time



H = 182 Oe

Number of flux lines in pickup loop = $AH/\varphi_0 = 10^8$

1/f => logarithmic decay of autocorrelation of N(t)

=> PECULIAR

How to derive macro from micro ?

- Rigorous first principle derivation normally impossible
- Use conservation laws, symmetries and intuition

Example:

- density fluctuations in an ensemble of particles

Choose:

- simple situation
- experimental relevant
- paradigmatic

VOLUME 64 25 JUNE 1990 NUMBER 26

Lattice Gas as a Model of $1/f$ Noise

Henrik Jeldtoft Jensen
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(Received 26 June 1989)

Direct numerical measurement of the power spectrum of the number of particles on the lattice demonstrates in an example of probably broad physical relevance that $1/f$ behavior can arise due to self-organized criticality. Different versions of the model are studied in order to look for universality.

Physica Scripta. Vol. 43, 593–595, 1991.

$1/f$ Noise from the Linear Diffusion Equation

Henrik Jeldtoft Jensen
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

Received November 9, 1990; accepted December 2, 1990

Abstract

It is pointed out that the ordinary linear diffusion equation $\partial P(r, t)/\partial t = \gamma\nabla^2 P(r, t)$, without any source term, leads to a $1/f$ power spectrum in any spatial dimension if the system is driven by a white noise *boundary* condition.

Nonlinearities and noise

PHYSICAL REVIEW A VOLUME 45, NUMBER 2 15 JANUARY 1992

$1/f^\alpha$ noise in dissipative transport

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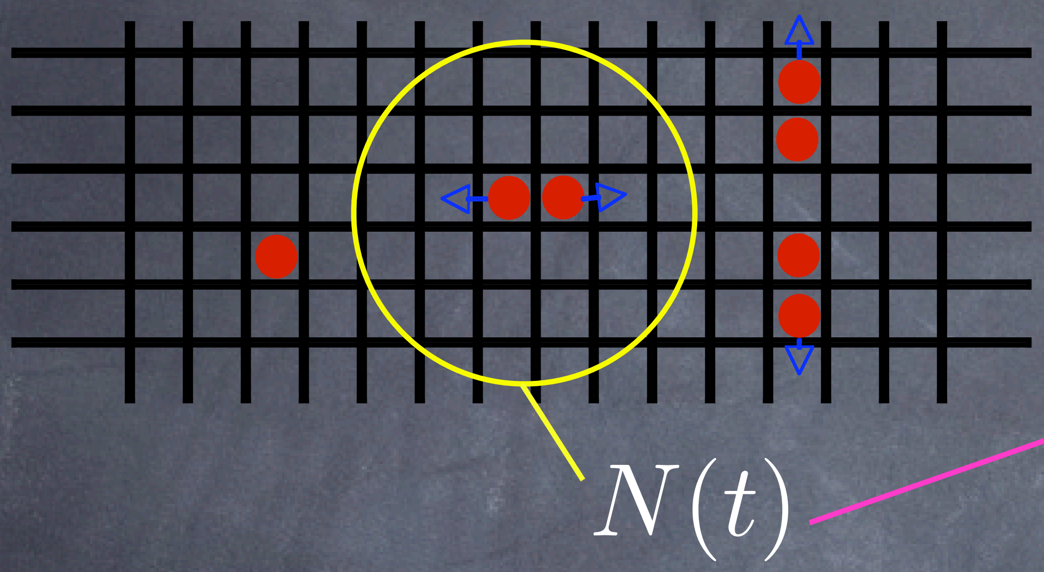
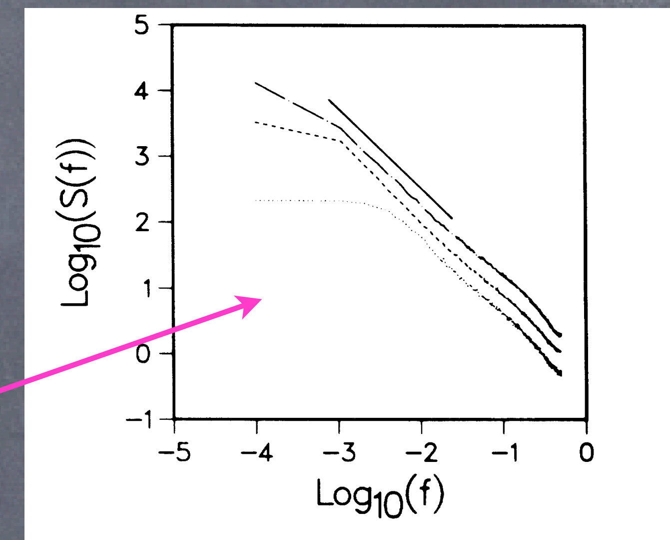
Henrik Jeldtoft Jensen
Nordisk Institut for Teoretisk Atomfysik, Blegdamsvej 17, DK-2100 Copenhagen ø, Denmark
(Received 8 March 1991)

Motivated by the hypothesis that "self-organized criticality" is a common source of $1/f$ noise, we construct and analyze a class of nonlinear nonequilibrium models describing the dissipative dynamics of interacting particles injected stochastically at the system boundaries. We show that such noisy boundary problems may be analyzed by renormalization-group methods and find that the noise spectrum for the particle number is $1/f$ in all dimensions in the absence of an external driving force or noise. Addition of such a force or of bulk noise changes the spectrum to $1/f^2$, or $1/f^{3/2}$, respectively. These results explain several recent numerical experiments on dissipative transport.

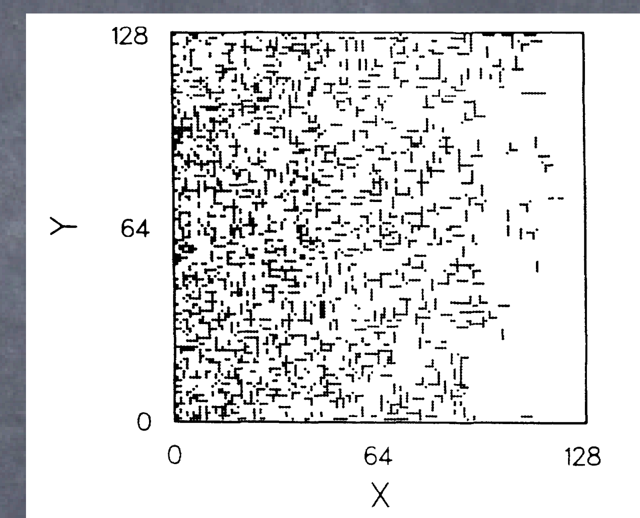
Deterministic lattice gas



1/f power spec



Dissipation occurs on a fractal



From HJ Jensen PRL 64,1 (1990)

Experiment on fluctuations in
vortex density in thin film
Yeh & Kao, PRL, 53, 1590 (1984)

Size dependence

Lattice gas

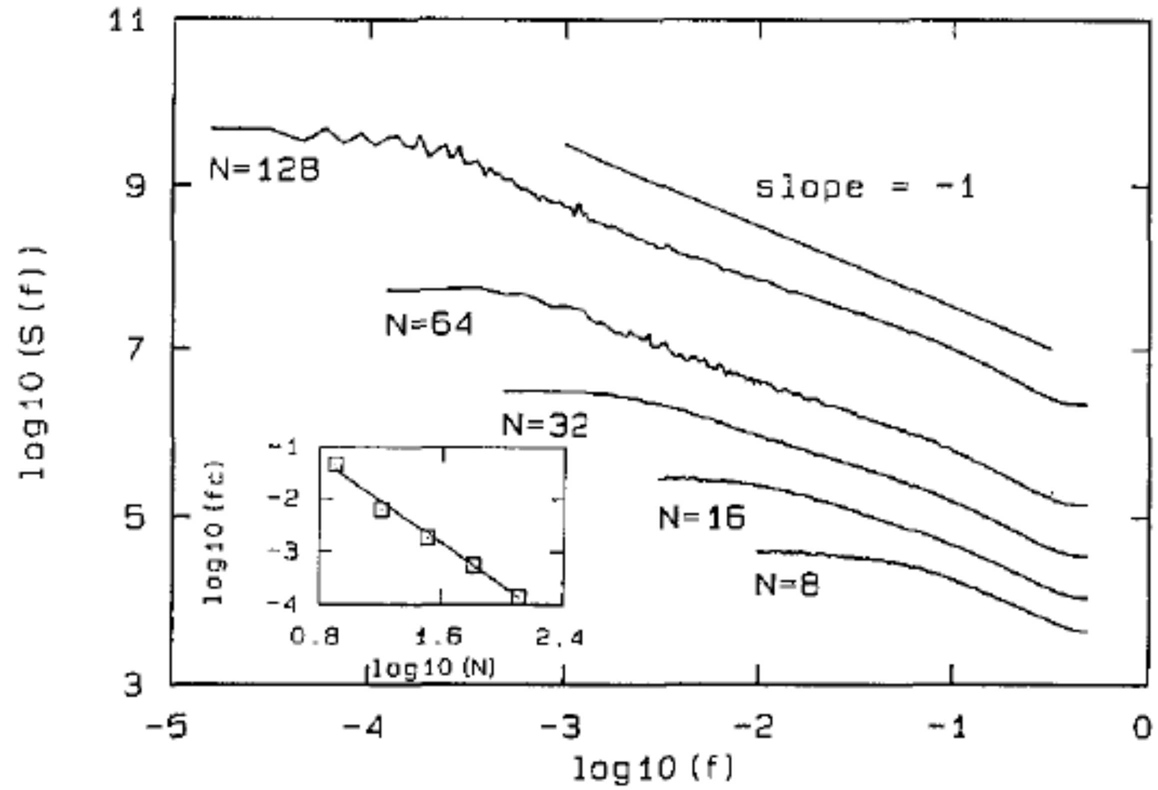
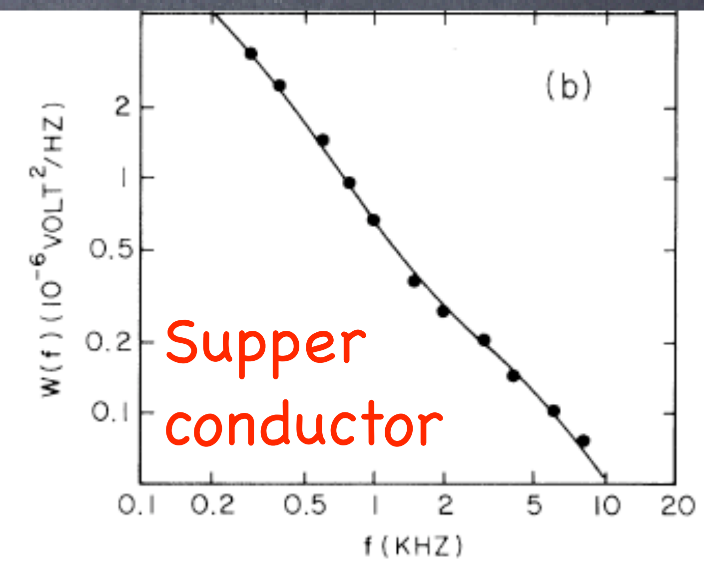


Fig. 4. The power spectrum $S(f)$ for the deterministic lattice gas model for different lattice sizes. It is seen to satisfy algebraic scaling with exponent $\beta = 1$. From the inset it follows that the crossover frequency f_c scales inversely with the volume of the system. The boundary drive is fixed to $p = 10^{-1}$ in all simulations.



Supper conductor

T. Fiig and H. J. Jensen, *Diffusive description of lattice gas models*. J. Stat. Phys. **71** 653 (1993)

Density dependence

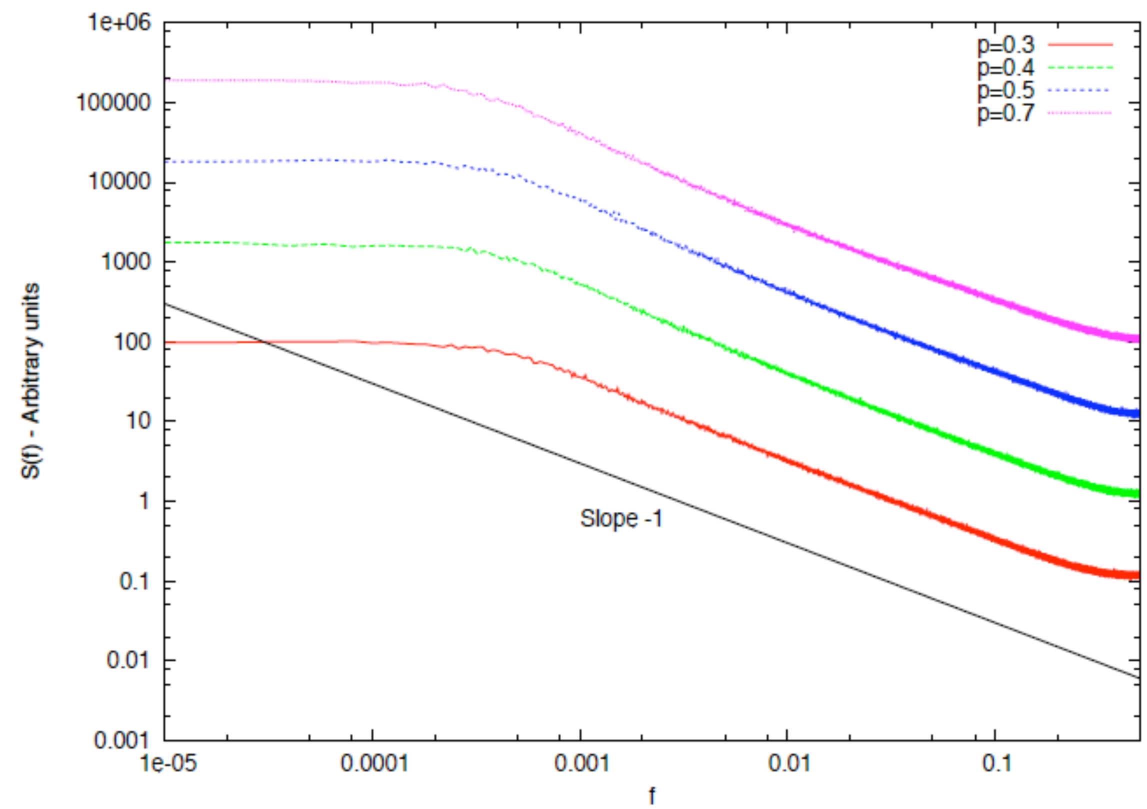


FIG. 12: Scaling behaviour of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles $N(t)$ in the dDLG for different boundary drives p and particles densities ρ. $S(f)$ has been multiplied by different constants for different drives p to visualise the scaling exponents properly. With small lattice sizes one observes scaling with $\beta = 1$. Lattice size $L = 64$.

From Master thesis
Andrea Giomette

Individual particles behave as random walkers

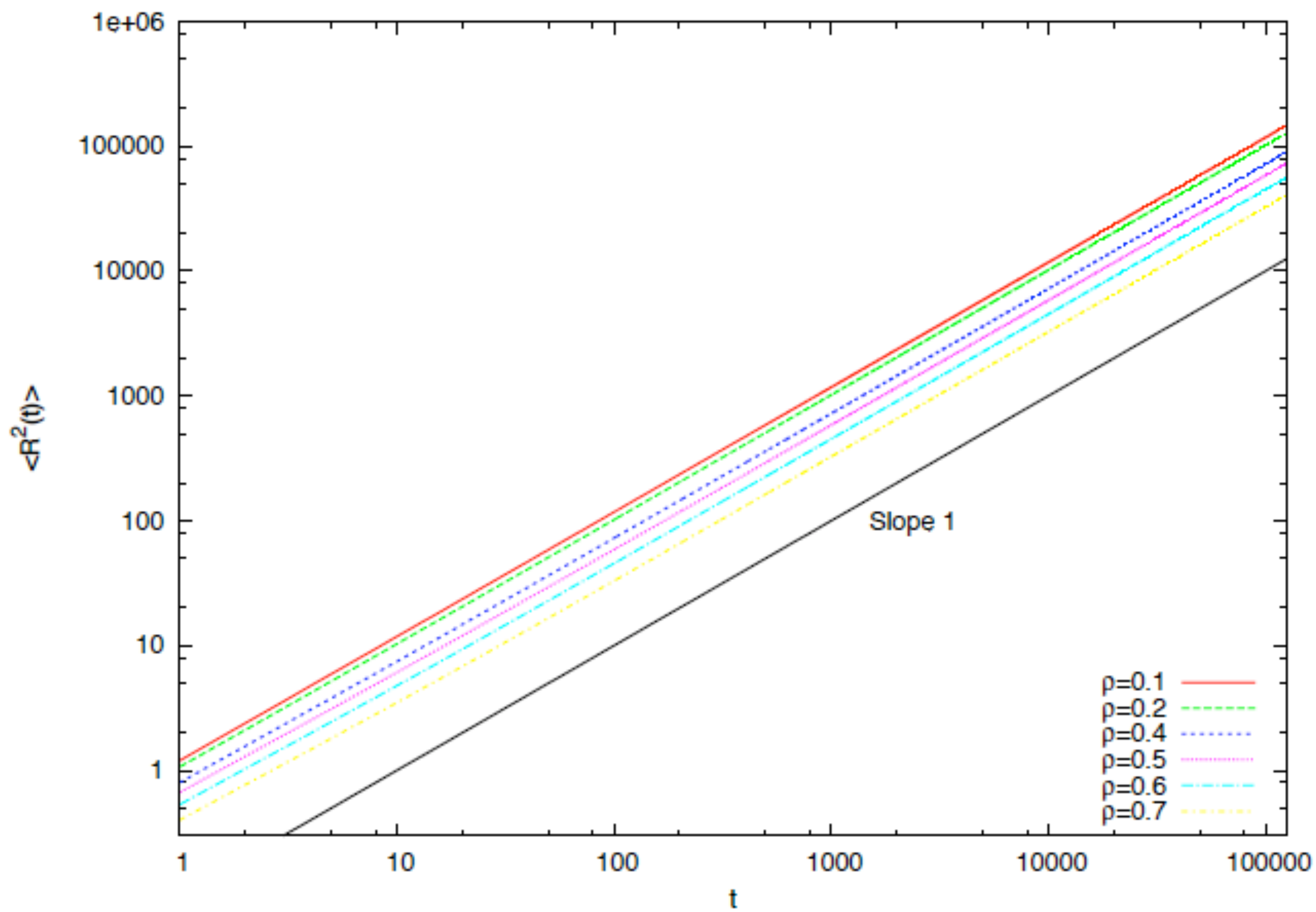


Figure 2.3: RW: Mean square displacement $\langle R^2(t) \rangle$ versus time t for different particles densities. Individual particles experience ordinary random walks. $L = 250$.

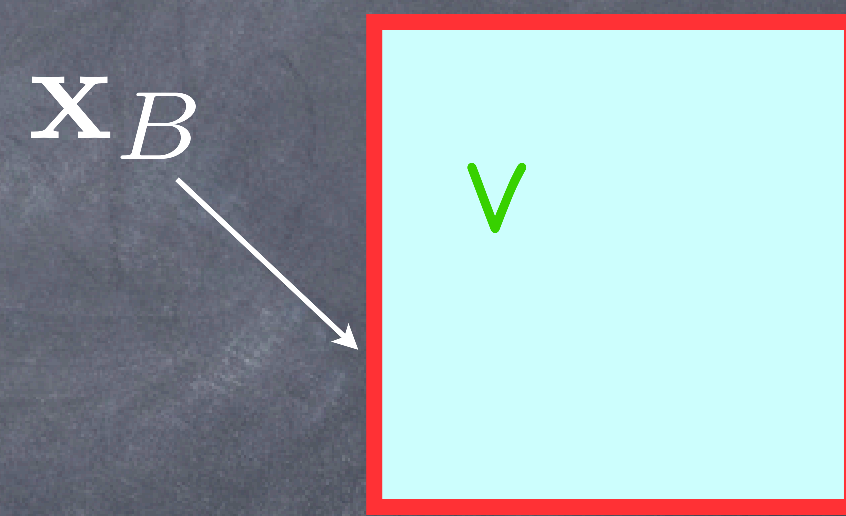
Understanding the behaviour

1/f from diffusion

$$\frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t) \quad \text{and} \quad n(\mathbf{x}_B,t) = \eta(\mathbf{x}_0,t)$$

then $N(t) = \int_V d\mathbf{x} n(\mathbf{x},t)$

is 1/f in any dimension.



See e.g.

- Grinstein, Hwa & Jensen, Phys. Rev. A **45** R559 (1992)
- H.J. Jensen, *Self-Organized Criticality*, Cambridge University Press 1998.

Effect of bulk noise

$$\frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t) + \nabla \cdot \vec{\chi}(\mathbf{x},t) \Rightarrow \mu = 3/2$$

or drive

$$S_N(\omega) \sim 1/\omega^{\mu}$$

drive breaks the symmetry x -> -x
and allows for a gradient term:

$$\frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t)) - a \frac{\partial n(\mathbf{x},t)}{\partial x} \Rightarrow \mu = 2$$

See e.g.

- Grinstein, Hwa & Jensen, Phys. Rev. A **45** R559 (1992)
- H.J. Jensen, *Self-Organized Criticality*, Cambridge University Press 1998.

Nice!

So we seem to have an understanding of the Yeh & Kao experiment

Except....... Andrea Giomette ... looked at bigger systems

Even bigger systems

Lattice gas

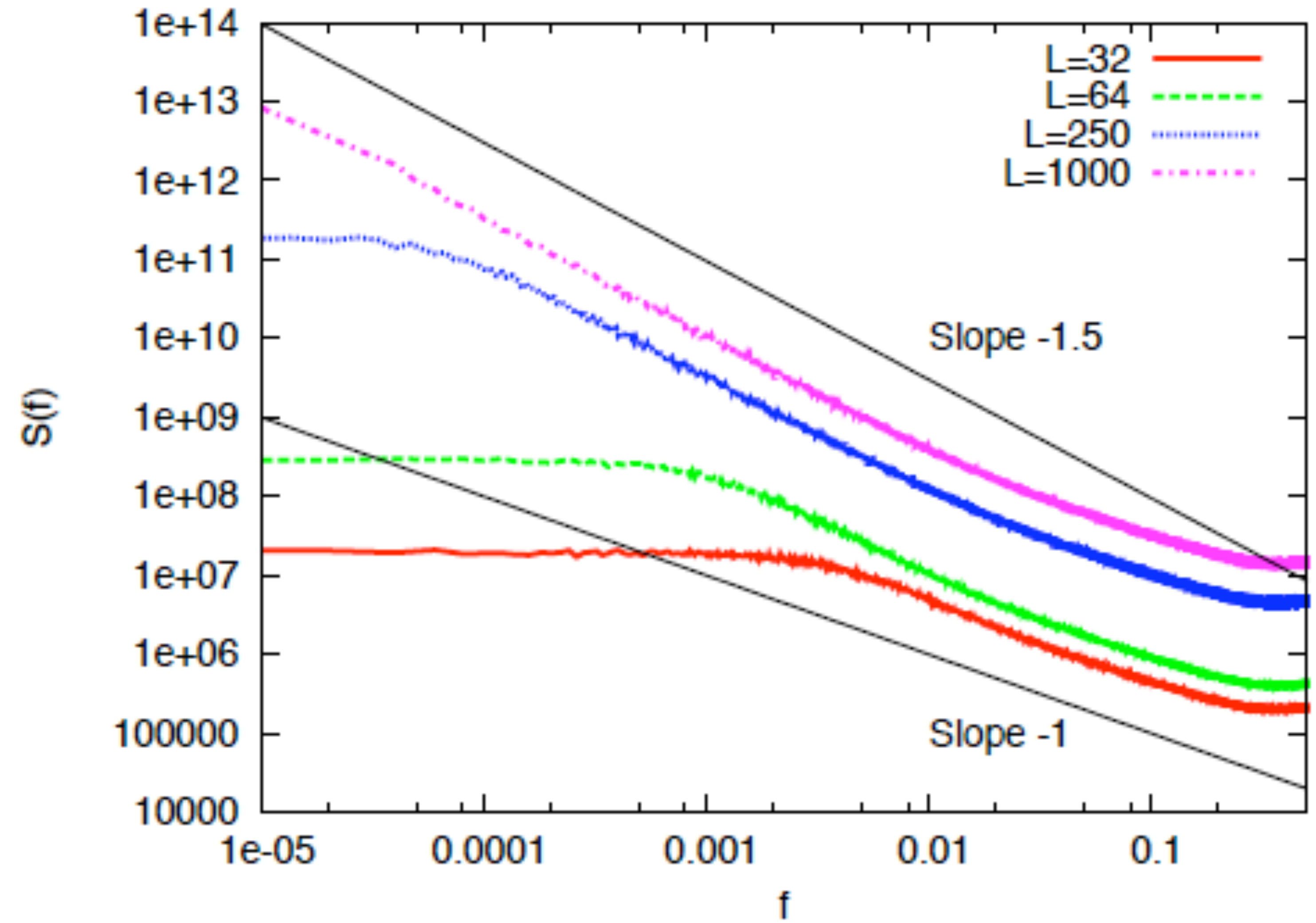


FIG. 1: Scaling behavior of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles $N(t)$ in the DLG for increasing linear sizes of the lattice L. A crossover from $\mu = 1$ for small L to $\mu = 1.5$ for large L is observed. Particle density $\rho = 0.5$.

From Master thesis
Andrea Giomette

Even bigger systems
- and density dependence

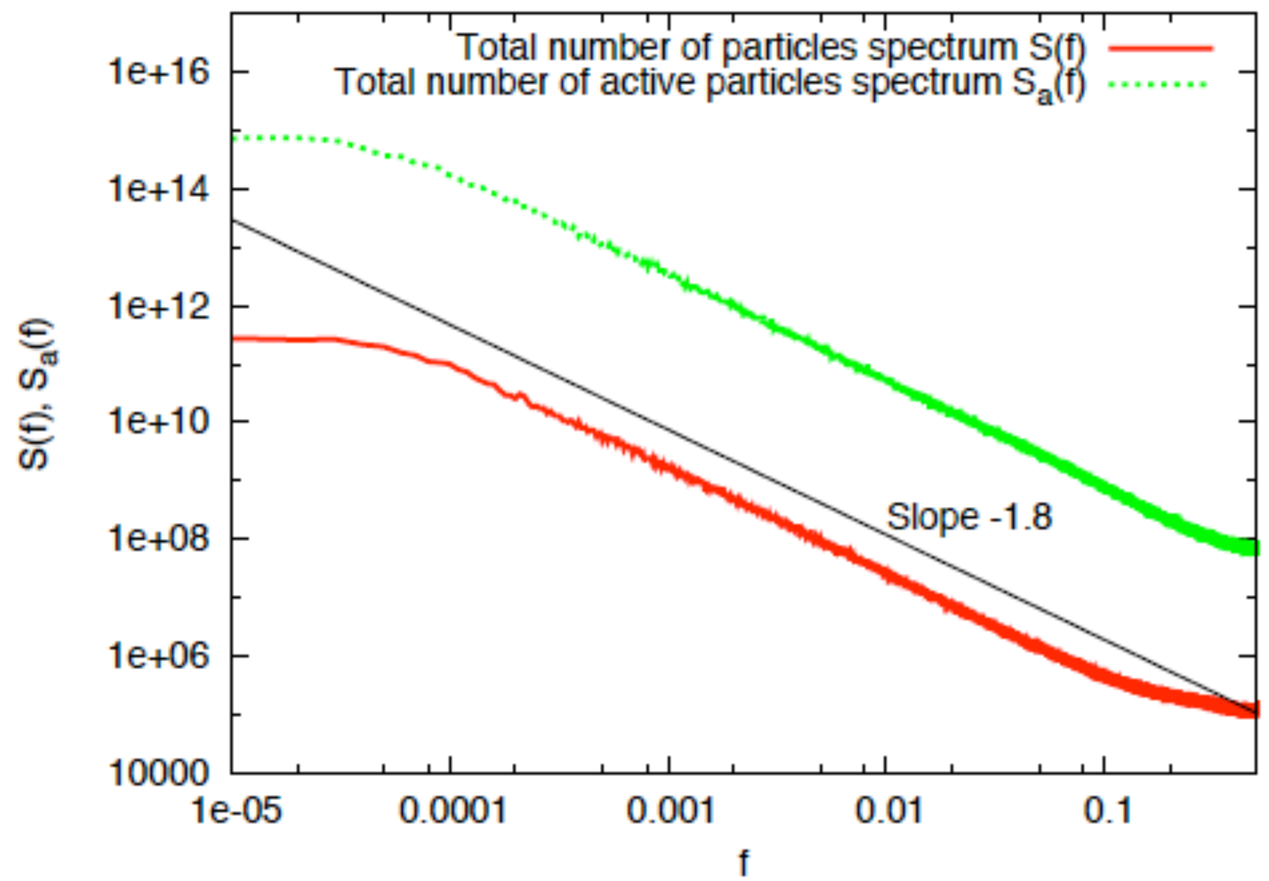


FIG. 10: DLG: Scaling behavior of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles in the box $N(t)$ and of the spectrum $S_a(f)$ of the total number of active particles near the critical density $\rho = 0.24506 \simeq \rho_c$. The spectra have been multiplied by arbitrary factors to visualize the scaling exponent properly. Lattice linear size $L = 1000$.

From Master thesis
Andrea Giomette

Larger systems

> at higher densities => μ = 3/2

> at low density => μ = 1.8

Explanation The β = 3/2 at high density

As the system increases a bulk noise term is generated

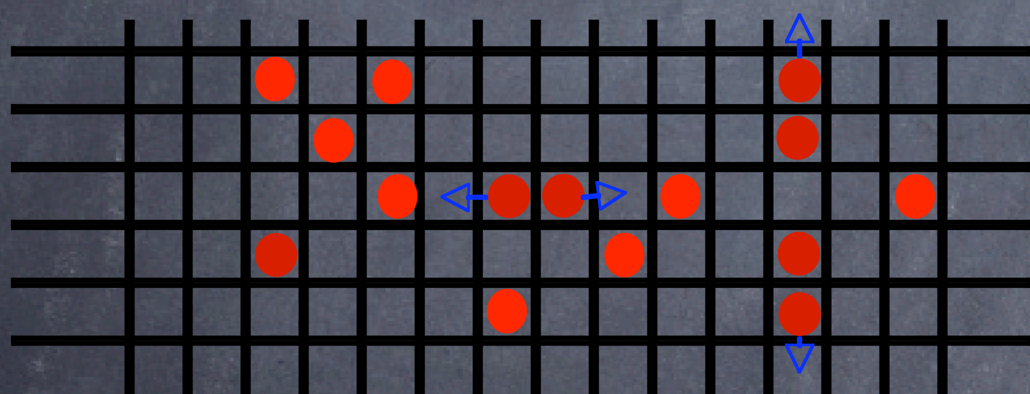
$$\frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t) \quad \longrightarrow \quad \frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t) + \nabla \cdot \vec{\chi}(\mathbf{x},t)$$

$$\mu = 3/2$$

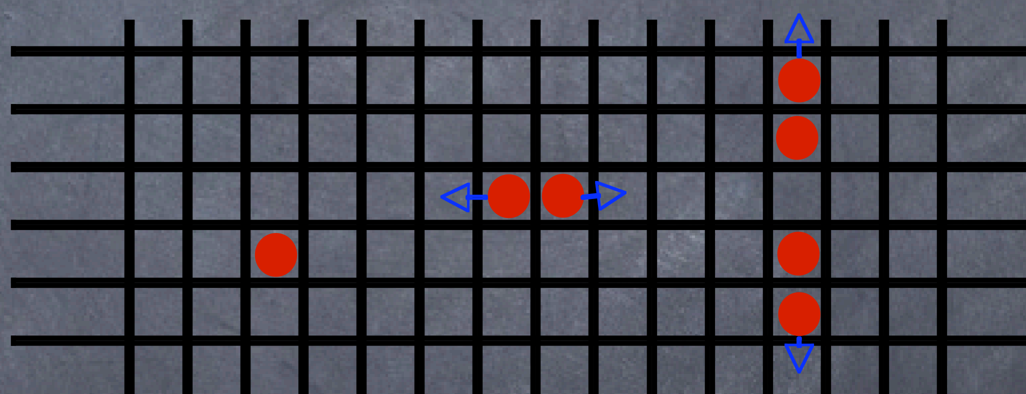
Explanation The $\beta = 1.8$ at low density

The absorbing state phase transition:

> as density decreases motion will stop

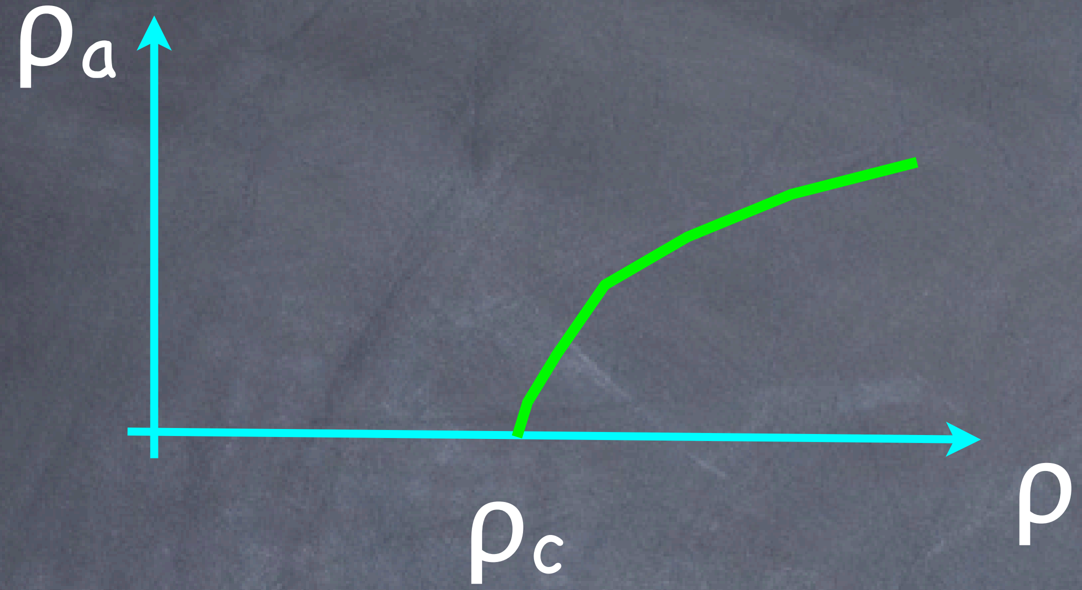


High density



Low density

The absorbing state phase transition



Density of active particles $$\rho_a \quad \sim \quad (\rho - \rho_c)^\beta$$

Survival probability $$P_\infty \quad \sim \quad (\rho - \rho_c)^{\beta'}$$

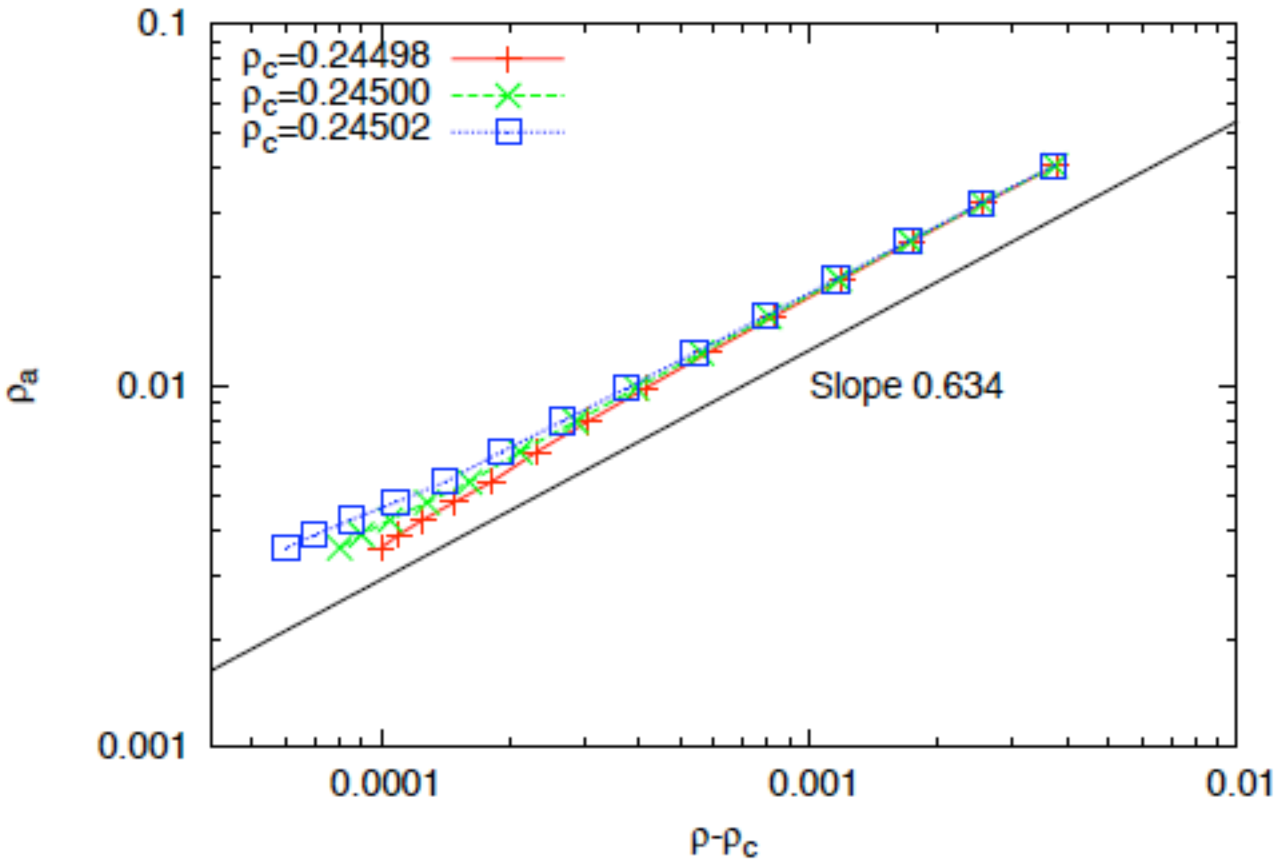


FIG. 3: Density of active sites ρ_a as a function of $\delta\rho = \rho - \rho_c$. The determination of the critical density ρ_c is obtained varying its value until data points are aligned on a straight line in a log-log plot. Lattice linear size $L = 1000$. Error bars are smaller than symbols.

$\beta = 0.634$

Study all the critical exponents

Determine universality class

	β	$\nu_\perp$	$\nu_\parallel$	σ
Manna	0.639(9)	0.799(14)	1.225(29)	2.229(32)
DLG	0.634(2)	0.83(5)	1.2(1)	2.19(1)
	γ'	γ	α	z
Manna	0.367(19)	1.590(33)	0.419(15)	1.533(24)
DLG	0.37(1)	1.54(1)	0.41(1)	1.5(1)

TABLE I: The measured critical exponents for the DLG and the corresponding critical exponents for the Manna universality class in $d = 2$[12].

Conclude:

Deterministic Lattice Gas belongs to the Manna universality class.

Hitherto unexpected because because all other members are stochastic models.

Back to power spectrum

$$\mu = 1 + \frac{1}{z}(2 - \frac{\beta}{\nu_\perp})$$

$$\beta = 0.643$$

$$z = 1.5$$

$$\nu_\perp = 0.83$$

$$\mu = 1.78(2)$$

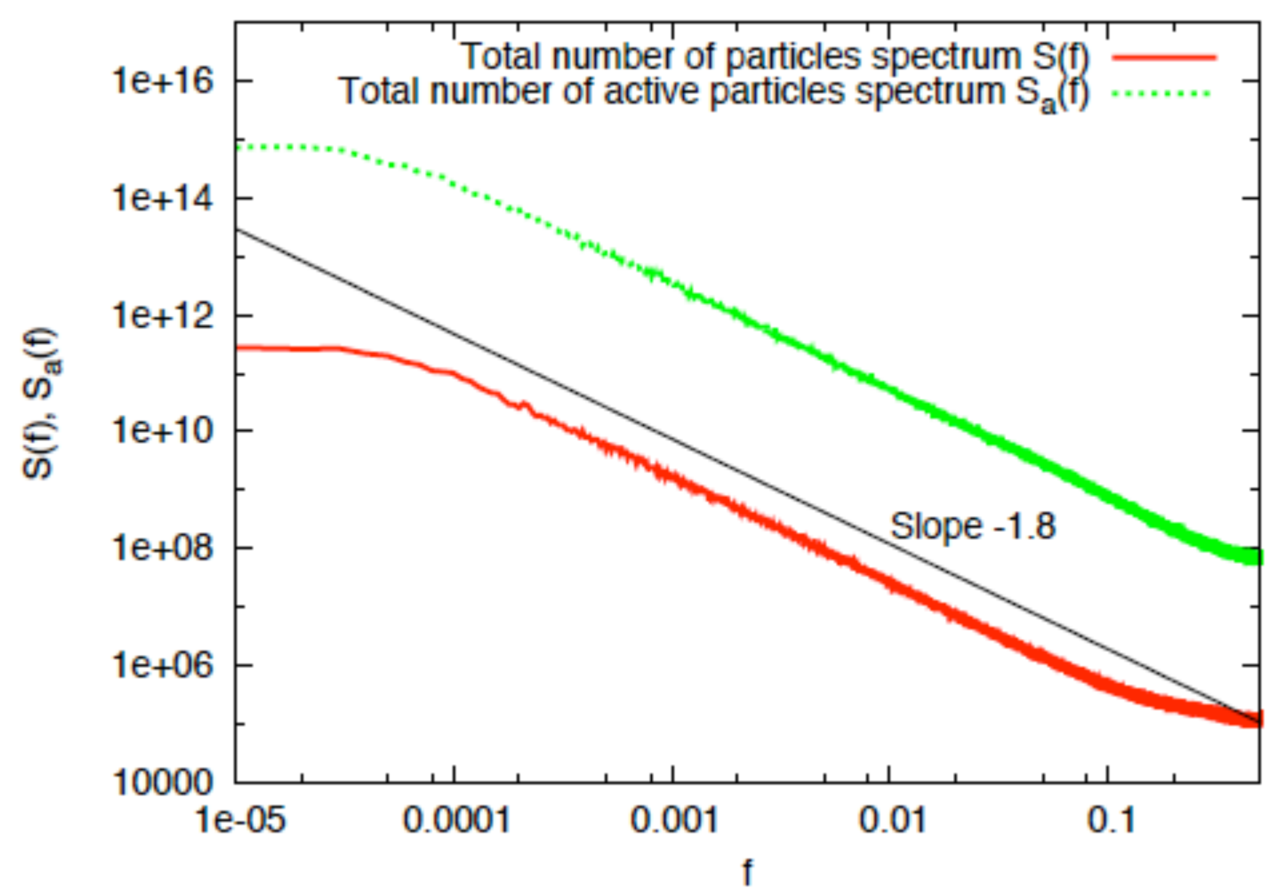


FIG. 10: DLG: Scaling behavior of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles in the box $N(t)$ and of the spectrum $S_a(f)$ of the total number of active particles near the critical density $\rho = 0.24506 \simeq \rho_c$. The spectra have been multiplied by arbitrary factors to visualize the scaling exponent properly. Lattice linear size $L = 1000$.

$\rho \approx \rho_a$ near transition

Summary of lattice gas behaviour:

> at elevated densities

phenomenology well described by Langevin Eq. with bulk noise

> at low densities

near the frozen state critical properties describes the fluctuations

Real systems

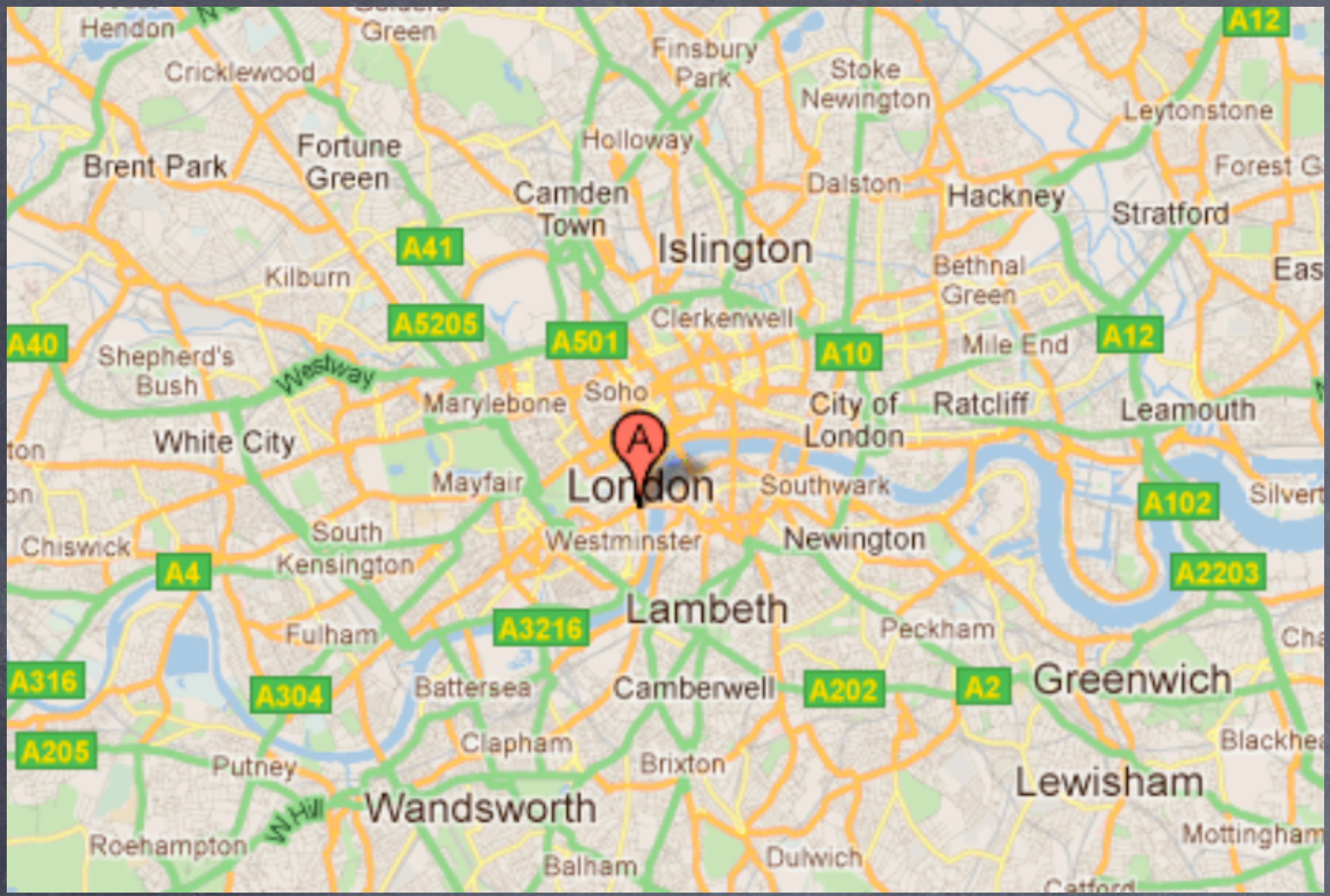
> Yeh & Kao experiment

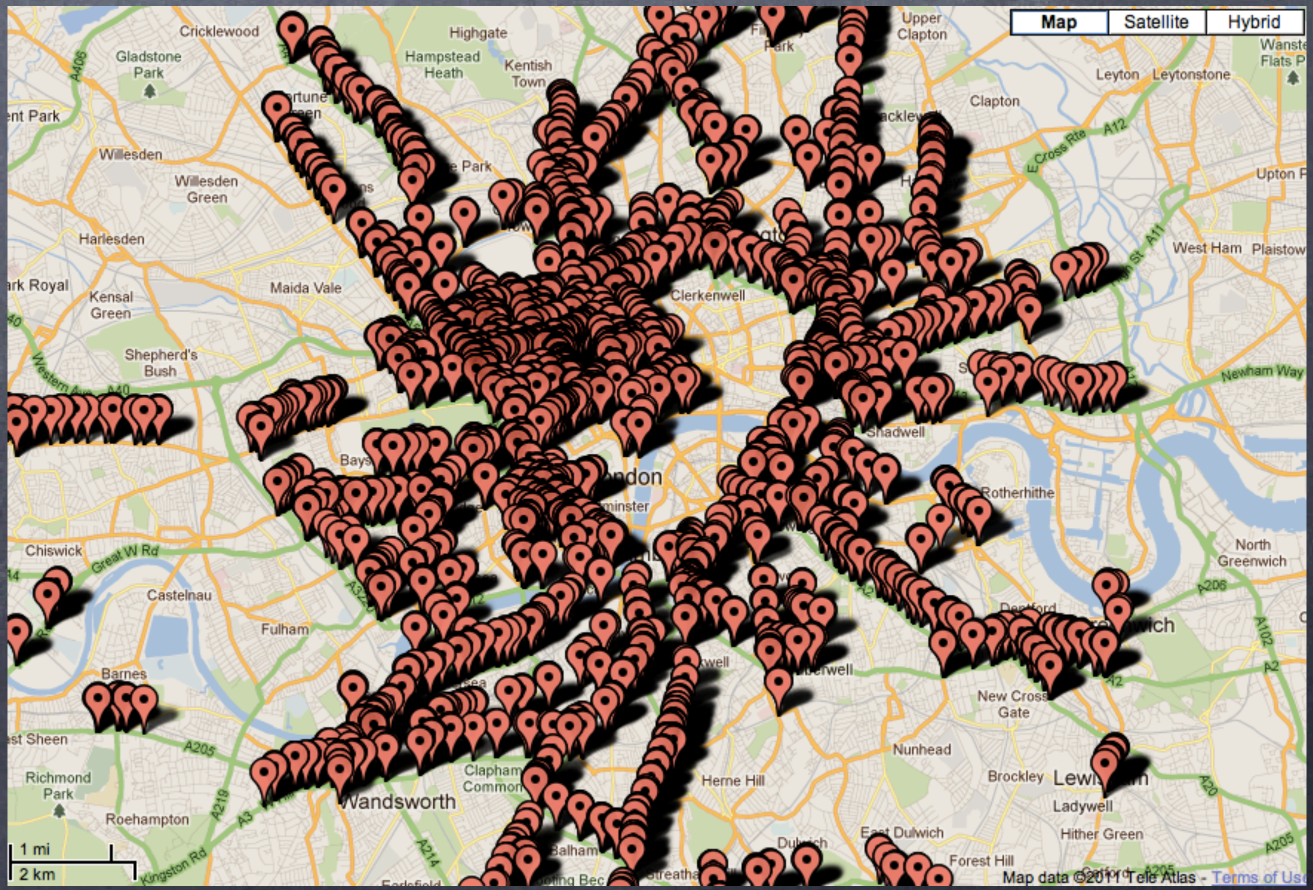
- not very many flux lines in the pick-up loop, so perhaps small size limit apply

- flux lines big and strongly interacting, so perhaps no bulk noises

> London traffic

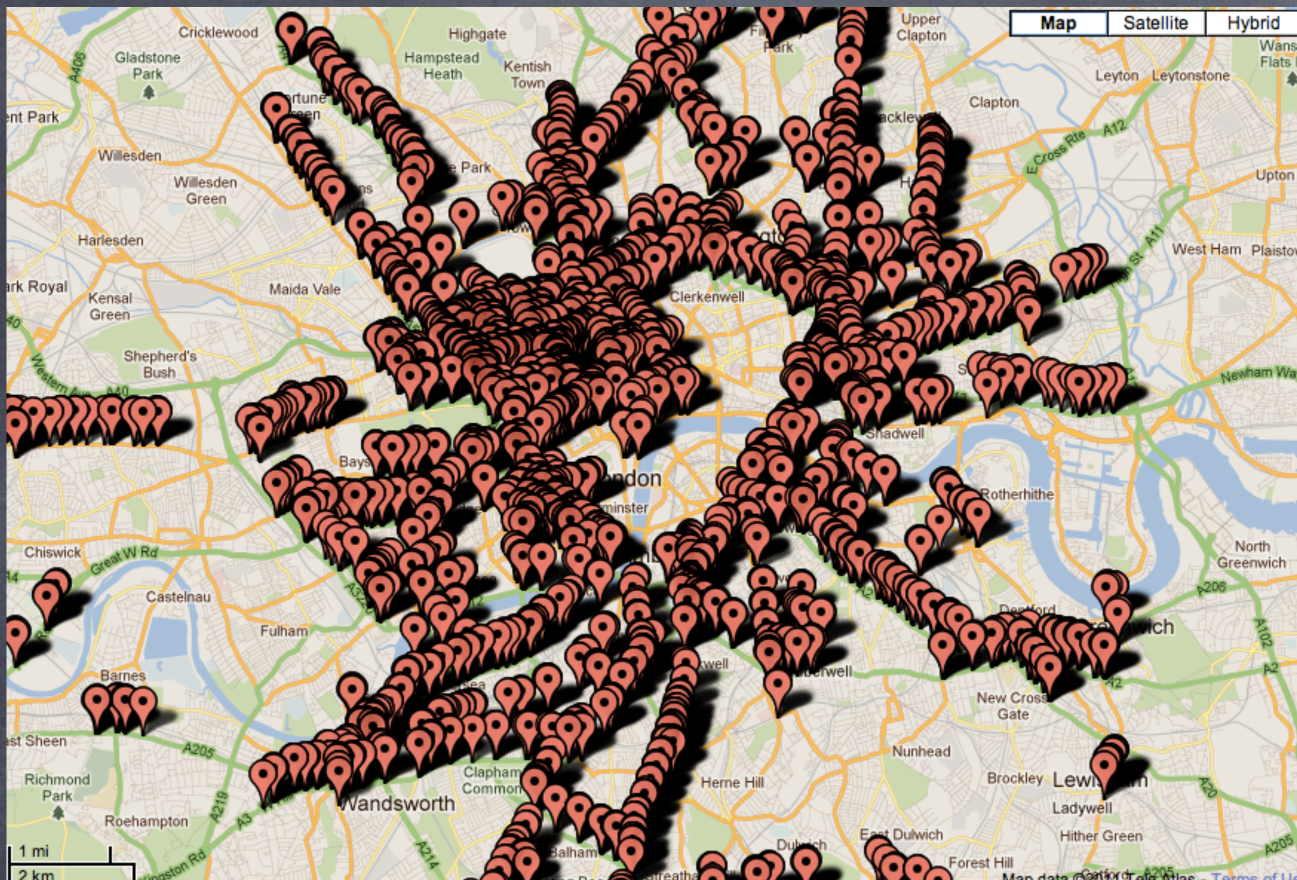
London traffic system - one big community





From Giovanni Petri & Paul Expert

London traffic system - one big community



Blue: one large community

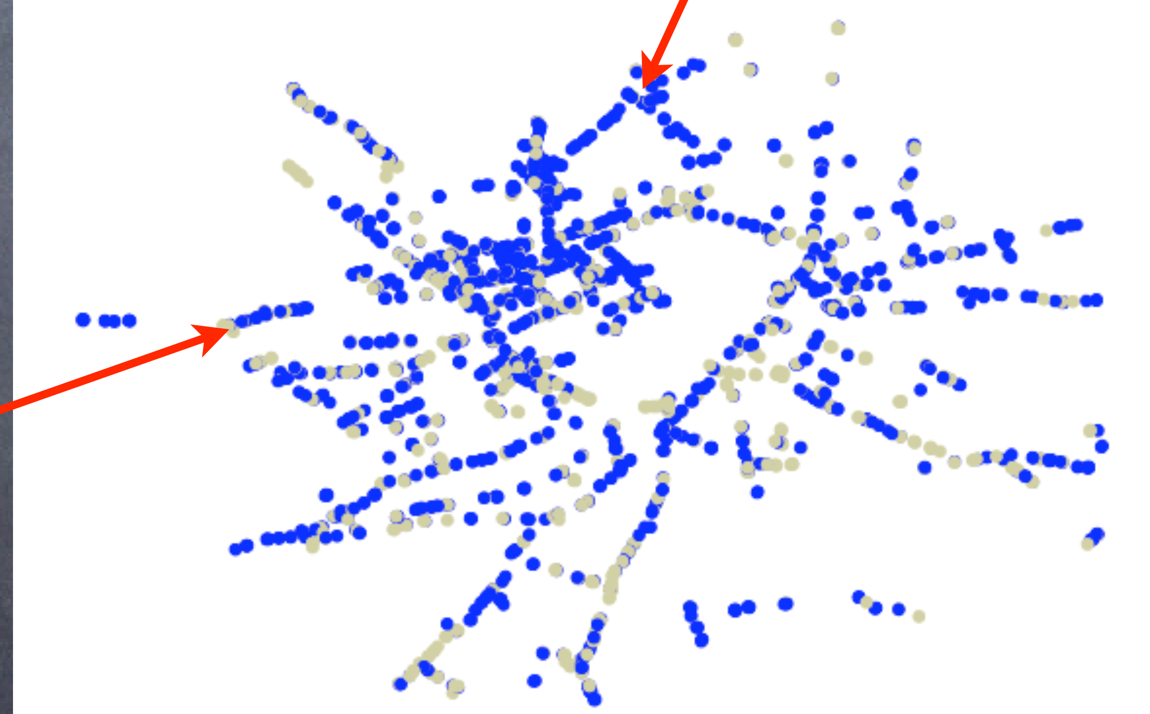


Fig. 1. A representation of the dynamical community structure of London's traffic. The blue dots belong to the largest community, that percolates through the whole network. The implication is that the whole network is a single large, highly correlated entity. The remaining communities have an average size of 2-3 spatially close sensors and have been grayed out for visibility.

Grey: many small (2-3 sensors) communities

From Giovanni Petri & Paul Expert

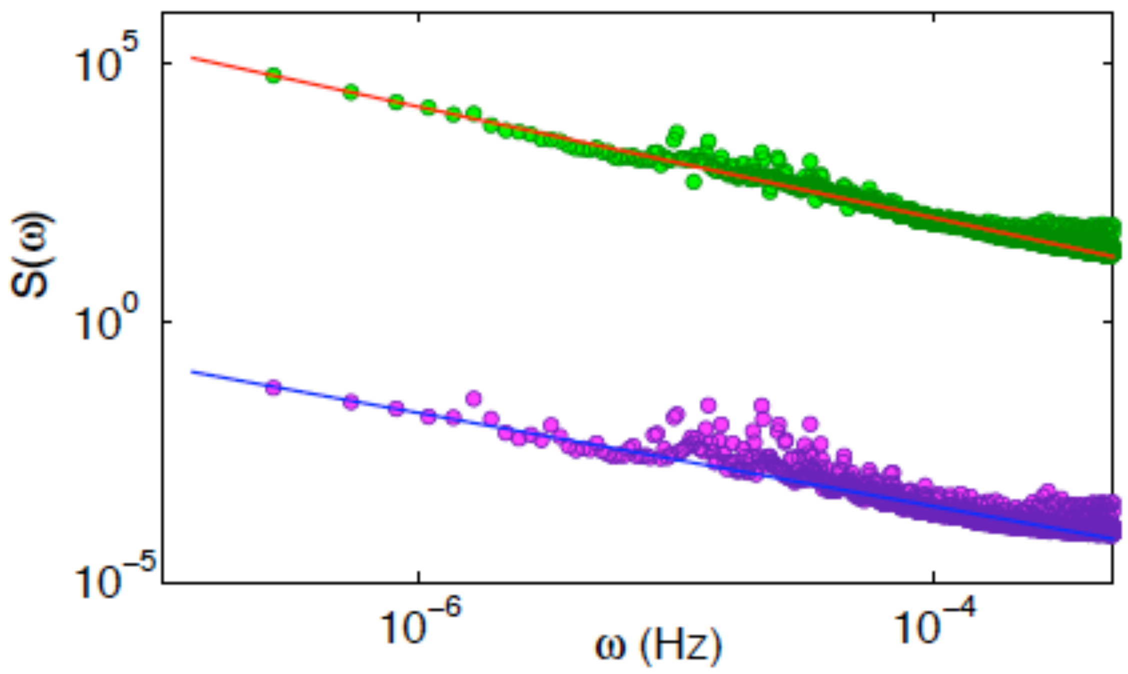


Fig. 2. Power spectra for the traffic flow (purple) and density (green), showing very clear power-law scaling behaviour with exponent $\alpha_f \simeq -0.9$ and $\alpha_o \simeq -1$ respectively. The long-range memory effects span a wide range of temporal scales, persisting over weeks. The observable peak at $\omega_{day} \simeq 10^{-5} Hz$ corresponds to the daily correlations. The flow data have been lowered for visibility.

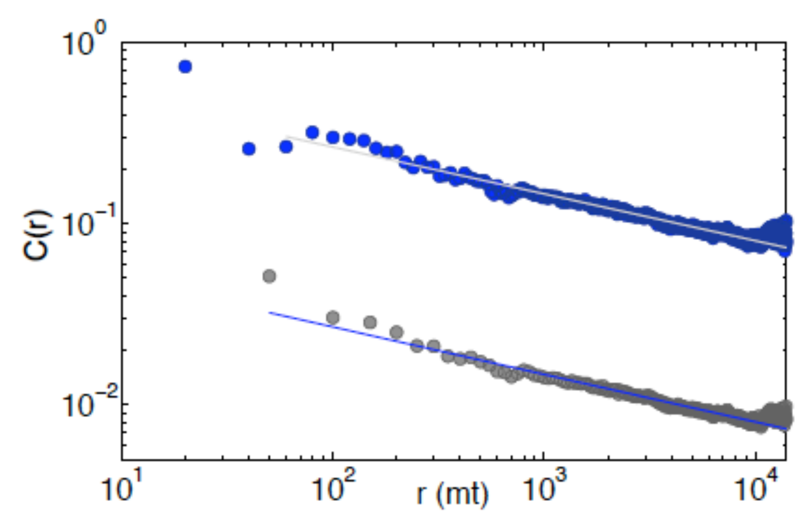


Fig. 3. Plot of the spatial correlations $C(r, \tau)$ as function of the spatial distance r for delay $\tau = 0$ (blue) and $\tau = 30$ minutes (gray). The slope is $\beta = -0.26 \pm 0.01$ implying a very slow decay of correlations over a large interval of distances ($10^2 - 10^4$ meters). The correlations for $\tau = 30$ minutes have been lowered for visibility, since the the two curves shown remarkable overlap.

From Giovanni Petri & Paul Expert

Conclusion

Going from micro to macro non trivial

The macroscopic Langevin Equation

Deterministic terms controlled by symmetries and conservation

Noise terms limited by symmetries and conservation - but not uniquely determined

Thank you

http://www2.imperial.ac.uk/~hjjens/

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