Identifying Self-Organised Criticality in nature
A guide by the confused

Gunnar Pruessner

Department of Mathematics
Imperial College London

Bern (ISSI meeting), September 2013
Outline

1. Definition of SOC
2. The (field) theory of SOC
3. Observables and Analysis
4. Summary
What is SOC?

As far as general use goes, what does SOC normally refer to?

Two extremes:

- Anything where “critical behaviour” is observed without tuning of a parameter.
- Anything avalanching.
What is SOC?
Critical behaviour without tuning?

Typical criticism:

- Is the Ising Model at $T = T_c$ SOC?
- Is percolation SOC ($p_c = 1/2$ for square, bond and triangular, site)?
- Is a fair random walker SOC?
- Is a fair branching process SOC?
- Is turbulence SOC?
What is SOC?
A footnote on turbulence

- Scaling largely a matter of dimensional analysis (trivial?)
- Separation of time scales in “output” rather than driving (Grinstein, 1995)
- Flow of energy to *smaller and smaller* length scales.
- Definition of avalanches only via explicit thresholding (not those of the dynamics)
What is SOC?

Anything avalanching?

Hoffmann, 2005, Figs. 9 and 10

- Wars (Roberts & Turcotte, 1998)
- Pop charts (Bentley & Maschner, 1999)
- Urban Development (Batty & Xie, 1999)
- Hospital waiting times (Smethurst & Williams, 2001)
- Avalanches of social norms (Hoffmann, 2005)
What is SOC?

Anything avalanching?

Smethurst & Williams, 2001, Fig. 1

- Wars (Roberts & Turcotte, 1998)
- Pop charts (Bentley & Maschner, 1999)
- Urban Development (Batty & Xie, 1999)
- Hospital waiting times (Smethurst & Williams, 2001)
- Avalanches of social norms (Hoffmann, 2005)
What is SOC?
Anything with a power law?

- Gravity, $F \propto r^{-2}$
- Hospital waiting times (Smethurst & Williams, 2001)
- Percolation
SOC

1. Non-trivial Scaling (finite size scaling — no control parameter)
2. Spatio-temporal correlations
3. Apparent self-tuning (underlying 2nd order phase transition?)
4. Separation of time scales
5. Avalanching (intermittency)
6. [nonlinear (thresholds) interaction] (supposedly required by 1)

SOC: Non-trivial scale invariance (spatio-temporal correlations!) in avalanching (intermittent) systems as known from ordinary critical phenomena, but with internal, self-organised rather than external tuning of a control parameter (to a non-trivial value).
The (field) theory of SOC

Stationarity is equivalent to self-organisation to critical point. Stationarity (lack of additional net deposition):

\[ \ldots = 0 \]

- Vanishing deposition at stationarity means that the diagrams in the bracket vanish.
- Requires adjustment of substrate.
- Independent of driving.
The (field) theory of SOC

Stationarity is equivalent to self-organisation to critical point. Stationarity (lack of additional net deposition):

\[
\times \left[ \begin{array}{c} \circ + \circ + \circ + \ldots \end{array} \right] \times \leftarrow = 0
\]

- Vanishing deposition at stationarity means that the diagrams in the bracket vanish.
- Requires adjustment of substrate.
- Independent of driving.
The (field) theory of SOC

Stationarity is equivalent to self-organisation to critical point.

Stationarity (lack of additional net deposition):

\[ \text{Vanishing deposition at stationarity means that the diagrams in the bracket vanish.} \]

\[ \text{Requires adjustment of substrate.} \]

\[ \text{Independent of driving.} \]
The (field) theory of SOC

Stationarity is equivalent to self-organisation to critical point.

Stationarity (lack of additional net deposition):

\[
\times \left[ \begin{array}{c}
\circ + \circ + \circ + \ldots
\end{array} \right] \times 0 = 0
\]

- Vanishing deposition at stationarity means that the diagrams in the bracket vanish.
- Requires adjustment of substrate.
- Independent of driving.
At or around criticality

Driving uniformly, at site 1, at site 0.

The Manna Model is at criticality: No hovering, no sweeping, no pinching.

Finite size scaling due to lowest mode $q_1 = \pi/L$. 
At or around criticality

System size $L = 128, L = 256, L = 512, L = 1024$.

The Manna Model is at criticality: No hovering, no sweeping, no pinching.

Finite size scaling due to lowest mode $q_1 = \pi/L$. 
Suitable observables

The substrate is a good place to look for self-organisation.

- The particle density adjusts, but its value is not universal (value to be compared to the same system).

- Correlations in the substrate may be absent or very weak. The occur to counter scaling in the dynamics.

The substrate is a bad place to look for criticality.
Suitable observables

The activity is a good place to look for scaling (integrated activity: avalanche metrics).

- Finite size scaling.
- Change of resolution.
- Thresholding? (may introduce spurious crossover)
- Block scaling (conditional to activity).
- Scaling should be compared to null models (is it just white noise?).
- Exponents are (supposedly) universal.
- Moment ratios are (supposedly) universal.
Block scaling

... is a form of subsampling.

- Change of system size may impossible (how about resolution, threshold — dangerous!).
- **Block finite size scaling:**
  Measure densities and fluctuations in varying box sizes.
Block scaling

... is a form of subsampling.

- Change of system size may impossible (how about resolution, threshold — dangerous!).
- **Block finite size scaling:**
  Measure densities and fluctuations in varying box sizes.
Block scaling

... is a form of subsampling.

- Change of system size may impossible (how about resolution, threshold — dangerous!).
- **Block finite size scaling:**
  Measure densities and fluctuations in varying box sizes.
Block scaling

... is a form of subsampling.

![Graph showing block scaling](image)

Contact process, Pruessner 2008, Fig. 1

- Change of system size may impossible (how about resolution, threshold — dangerous!).
- **Block finite size scaling:**
  Measure densities and fluctuations in varying box sizes.
Block scaling

... is a form of subsampling.

Contact process, Pruessner 2008, Fig. 2

- Change of system size may impossible (how about resolution, threshold — dangerous!).
- Block finite size scaling:
  Measure densities and fluctuations in varying box sizes.
Summary

- A solid definition of SOC is hard to come by.
- I propose: Scaling (non-trivial, spatio-temporal, finite size), self-organisation to a critical point, intermittency, non-linear interaction.
- Henrik Jensen: SDIDT (slowly driven, interaction dominated, threshold systems).
- Field theory: Truly *at* the critical point.
- Observables: Scaling to be found in the activity, not the substrate.
- Block scaling?

THANKS!

Supported by the ISSI and EPSRC platform grant EP/I019111/1.