Phase mixing of kink waves in solar flux tubes

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Transverse oscillations of coronal loops

- First observed with *TRACE* in 1999
  Nakariakov et al. (1999); Aschwanden et al. (1999)
- After an energetic disturbance (e.g., a flare), the whole loop displays a damped transverse oscillation

Image credit: E. Verwichte

Nakariakov et al. (1999)
Theoretical interpretation

1. Quasi-mode
   - Transverse oscillation is the MHD Kink mode
e.g., Edwin & Roberts (1983)
   - Damping due to resonant absorption in the Alfvén continuum
e.g., Ruderman & Roberts (2002); Goossens et al. (2002)

2. Spatial Fourier Expansion
   - The superposition of Alfvén continuum modes builds up the global kink motion
   - The damping of the global motion is caused by the phase mixing of the Alfvén modes
e.g., Cally (1991); Soler & Terradas (2015)

3. Time-dependent numerical simulations
   - Full temporal behavior
   - Complicated effects: second order, nonlinear, etc…
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Quasi-mode
Homogeneous magnetic cylinder

Edwin & Roberts (1983)
The kink mode of a homogeneous tube

- Global transverse motion of the flux tube
- No damping

Thin tube

$(L/R \gg 1)$

approximation:

$$P = \frac{2L}{v_{A,i}} \sqrt{\frac{\rho_i + \rho_e}{2\rho_i}}$$
Transversely non-uniform cylinder

- A nonuniform layer is added

- $l = 0 \rightarrow$ Abrupt density jump
- $l = 2R \rightarrow$ Fully nonuniform tube
The quasi-mode

Assumptions

1. The kink mode still exists as a global mode (quasi-mode) \( \sim \exp(-i\omega t) \)
2. The kink mode is resonant in the Alfvén continuum

Diagram:

- Loop core
- Transition
- Exterior

- \( \omega_{\text{kink}} \)
- \( \omega_{A,i} \)
- \( \omega_{A,e} \)
The quasi-mode

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   \[ \sim \exp(-i\omega t) \]
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![Diagram showing the transition between Loop core, Transition, and Exterior regions, with \( \omega_{kink} \), \( \omega_{A,i} \), and \( \omega_{A,e} \) marked. The diagram indicates Alfvén resonance at the transition point.]
Computing the quasi-mode

1. Thin tube \((L/R \gg 1)\) + thin boundary \((l/R \ll 1)\) approximations
   (e.g., Ruderman & Roberts 2002; Goossens et al. 2002)
   - Correction to the kink mode of the homogeneous tube
   - Analytic expressions for \(P\) and \(\tau_D/P\)
   - TT is OK: observations typically show that \(L/R \sim 10^2\)
   - **There is no observational support for TB!**

2. Full solution of eigenvalue problem for arbitrary parameters
   - Resistive MHD eigenmode: Fully numerical
     (e.g., Van Doorsselaere et al. 2004; Arregui et al. 2005)
   - Ideal MHD quasi-mode: Semi-analytic
     (Soler et al. 2013, 2014)
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Thin tube \((L/R \gg 1)\), thin boundary \((l/R \ll 1)\) approximation

\[
P = \frac{2L}{\nu_{A,i}} \sqrt{\frac{(\rho_i + \rho_e)}{2\rho_i}}
\]

\[
\frac{\tau_D}{P} = F \frac{R \rho_i + \rho_e}{l \rho_i - \rho_e}
\]

- Linear: \(F = 4/\pi^2 \approx 0.405\)
- Sinusoidal: \(F = 2/\pi \approx 0.637\)
- Parabolic:
  \(F = 4\sqrt{2}/\pi^2 \approx 0.573\)
Beyond TTTB

- Governing equation for $P'$ in the $\beta = 0$ approximation

\[
\frac{\partial^2 P'}{\partial r^2} + \left[ \frac{1}{r} - \frac{\frac{d}{dr} \left( \rho(r) \left( \omega^2 - \omega_A^2(r) \right) \right)}{\rho(r) \left( \omega^2 - \omega_A^2(r) \right)} \right] \frac{\partial P'}{\partial r} + \left( \frac{\rho(r) \left( \omega^2 - \omega_A^2(r) \right)}{B^2/\mu} - \frac{m^2}{r^2} \right) P' = 0
\]

- General solution: $P'(r) = A_0 P'_1(r) + S_0 P'_2(r)$

- Frobenius series around the resonance position, $\omega = \omega_A(r_A)$

\[
P'_1(r) = (r - r_A)^2 \sum_{k=0}^{\infty} a_k (r - r_A)^k
\]

\[
P'_2(r) = \sum_{k=0}^{\infty} s_k (r - r_A)^k + CP'_1(r) \ln (r - r_A)
\]

\[
C = \frac{m^2}{2r_A^2}
\]
Quasi-mode frequency

- Sinusoidal transition

![Graph showing the relationship between $\omega_R/\omega_k$ and $l/R$ for different $\rho_i/\rho_e$ values: $\rho_i/\rho_e = 2$, $\rho_i/\rho_e = 5$, $\rho_i/\rho_e = 10$, and $\rho_i/\rho_e = 20$. The graph illustrates how the frequency ratio changes with the ratio $l/R$.](image-url)
Quasi-mode damping rate

- Sinusoidal transition

![Graph showing damping rate vs l/R for different \( \rho_i/\rho_e \) values. The graph plots \( \omega_i/\omega_k \) against l/R, with curves indicating different \( \rho_i/\rho_e \) values: 2, 5, 10, and 20. Each curve represents a different density ratio, with distinct markers and line styles for each.]
Error due to the TB approximation: $P$

- Weak dependence on $\rho_i/\rho_e$
- The larger $I/R$, the larger the error
- Some differences between density profiles
Error due to the TB approximation: $\tau_D/P$

- Weak dependence on $\rho_i/\rho_e$
- Complicated dependence on $l/R$
- Strong influence of the density profile!
Resonant absorption is an ideal process → Wave energy is conserved
Damping \neq\ Dissipation
Radial flux of energy

\[ \langle S_r \rangle = -\frac{1}{2} \text{Re} (i \omega \xi_r P'^*) \]
Linear numerical simulation

- Energy transfer from transverse (kink) motions to azimuthal (rotational) motions \( \rightarrow \) “Mode conversion”
- Phase mixing of rotational motions

Goossens et al. (2014)
Spatial Fourier Expansion
An alternative view

- Based on the paper *Phase mixing and surface waves: a new interpretation* by P. S. Cally, J. Plasma Physics 45, 1991
- Adapted to a cylindrical flux tube

- No global mode, no assumed coordinated motion of the flux tube
- Plasma motions described by the superposition of Alfvén continuum modes
- Full temporal evolution is retained

- Damping of the kink oscillation due to **phase mixing**
- Conversion from transverse to rotational motions consistently described
Method

- Linear incompressible MHD equations
- Compressibility of the kink mode (Goossens et al. 2009)

\[ \nabla \cdot \mathbf{v} \sim (k_z R)^2 \sim \left( \frac{R}{L} \right)^2 \]

- Governing equation for \( \xi_r \)

\[ \mathcal{L}_A \mathcal{L}_S \xi_r + \left( k_z^2 + \frac{m^2}{r^2} \right) \frac{\mathrm{d} \rho(r)}{\mathrm{d} r} \frac{\partial^2}{\partial t^2} \frac{1}{r} \frac{\partial (r \xi_r)}{\partial r} = 0 \]

- Alfvén wave operator:

\[ \mathcal{L}_A \equiv \rho(r) \frac{\partial^2}{\partial t^2} + \frac{B^2}{\mu} k_z^2 \]

- Surface wave operator:

\[ \mathcal{L}_S \equiv \left( k_z^2 + \frac{m^2}{r^2} \right) \frac{\partial^2}{\partial r^2} + \left( k_z^2 + \frac{3m^2}{r^2} \right) \frac{1}{r} \frac{\partial}{\partial r} - \left[ \left( k_z^2 + \frac{m^2}{r^2} \right)^2 + \left( k_z^2 - \frac{m^2}{r^2} \right) \frac{1}{r^2} \right] \]
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Solution in the uniform regions

- Decoupled Alfvén waves and surface waves
  \[ \mathcal{L}_A \xi_r = 0 \]
  \[ \mathcal{L}_S \xi_r = 0 \]

- \( m = 1 \) surface wave (kink mode)
- Internal plasma \( (r \leq R - \frac{l}{2}) \):
  \[ \xi_r(r, t) = A_i(t) l_1'(k_z r) \sim A_i(t) \text{ if } k_z R \ll 1 \]
- External plasma \( (r \geq R + \frac{l}{2}) \):
  \[ \xi_r(r, t) = A_e(t) K_1'(k_z r) \sim A_e(t) \frac{1}{r^2} \text{ if } k_z R \ll 1 \]
Solution in the uniform regions

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- External plasma (\( r \geq R + \frac{l}{2} \)):
  \[ \xi_r(r, t) = A_e(t) K'_1(k_z r) \sim A_e(t) \frac{1}{r^2} \quad \text{if} \quad k_z R \ll 1 \]
Solution in the nonuniform boundary

- Alfvén waves and surface waves are unavoidably coupled
- Generalized Fourier series

\[ \xi_r(r, t) = \sum_{n=1}^{\infty} a_n(t) \psi_n(r) \]

\[ a_n(t = 0) = \int_{R-l/2}^{R+l/2} \xi_r(r, t = 0) \psi_n(r) r \, dr \]

- Base functions \( \psi_n(r) \)

\[ \frac{d^2 \psi_n}{dr^2} + \frac{1}{r} \frac{d \psi_n}{dr} + \left( \lambda^2 - \frac{1}{r^2} \right) \psi_n = 0 \]

\[ \frac{d \psi_n}{dr} = 0 \quad \text{at} \quad r = R - \frac{l}{2} \]

\[ \frac{d \psi_n}{dr} + \frac{2}{r} \psi_n = 0 \quad \text{at} \quad r = R + \frac{l}{2} \]
**Alfvén continuum modes**

- Temporal evolution of coefficients $a_n(t) \sim \exp(-i\omega t)$
- Generalized eigenvalue problem: $\mathcal{H}a = \omega^2 M a$

\[
\mathcal{H}_{nn'} = k_z^2 \frac{B^2}{\mu} \frac{1}{l} \int_{R-l/2}^{R+l/2} \psi_n(r) \mathcal{L}_S \psi_{n'}(r) r dr
\]

\[
M_{nn'} = \frac{1}{l} \int_{R-l/2}^{R+l/2} \left[ \rho(r) \mathcal{L}_S \psi_{n'}(r) \right.
\]

\[
+ \frac{d\rho(r)}{dr} \left( k_z^2 + \frac{m^2}{r^2} \right) \left( \frac{d\psi_{n'}(r)}{dr} + \frac{1}{r} \psi_{n'}(r) \right) \left. \right] \psi_n(r) r dr
\]
Time-dependent solution

- Phase 1: Damping of the global transverse motion
- Phase 2: Motions become rotational in the nonuniform layer
Excellent agreement, specially for thin nonuniform layers

For thick layers, undamped oscillations after the global motion is damped
Flux of wave energy to the nonuniform boundary
Smaller and smaller spatial scales are generated as time increases.

Phase mixing length scale (Mann et al. 1995):

\[ L_{ph} = \frac{2\pi}{|\partial \omega_A / \partial r| t} \]
Energy cascade

\[ \xi_r(r, t) = \sum_{n=1}^{\infty} a_n(t) \psi_n(r) \rightarrow E \sim a_n^2(t) \]
Conclusion
Comparison

1. Quasi-mode
- The kink oscillation is understood as a global mode that is damped due to resonant absorption in the Alfvén continuum
- Only phase 1 (damping of global motion) is described
- Simple expressions for $P$ and $\tau_D$ are obtained in the TTTB approximation

2. Spatial Fourier Expansion
- Temporal evolution of kink oscillations built up as the superposition and phase mixing of Alfvén continuum modes
- Both phase 1 (damping of global motion) and phase 2 (generation of small-scale rotational motions) are described
- No simple expressions for $P$ and $\tau_D$ are obtained
1. Quasi-mode

- The kink oscillation is understood as a global mode that is damped due to **resonant absorption** in the Alfvén continuum.
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What have we learnt?

- Resonant absorption and phase mixing are **two aspects** of the same underlying physical process
- **Energy cascade from large scales to small scales**

Missing Ingredients

- Dissipation of the small scales
- Associated heating
- Nonlinearity: KHI can generate shorter scales and enhance dissipation and heating

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Final Remarks

What have we learnt?

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