



# Rossby waves: an introduction

Teimuraz Zaqarashvili

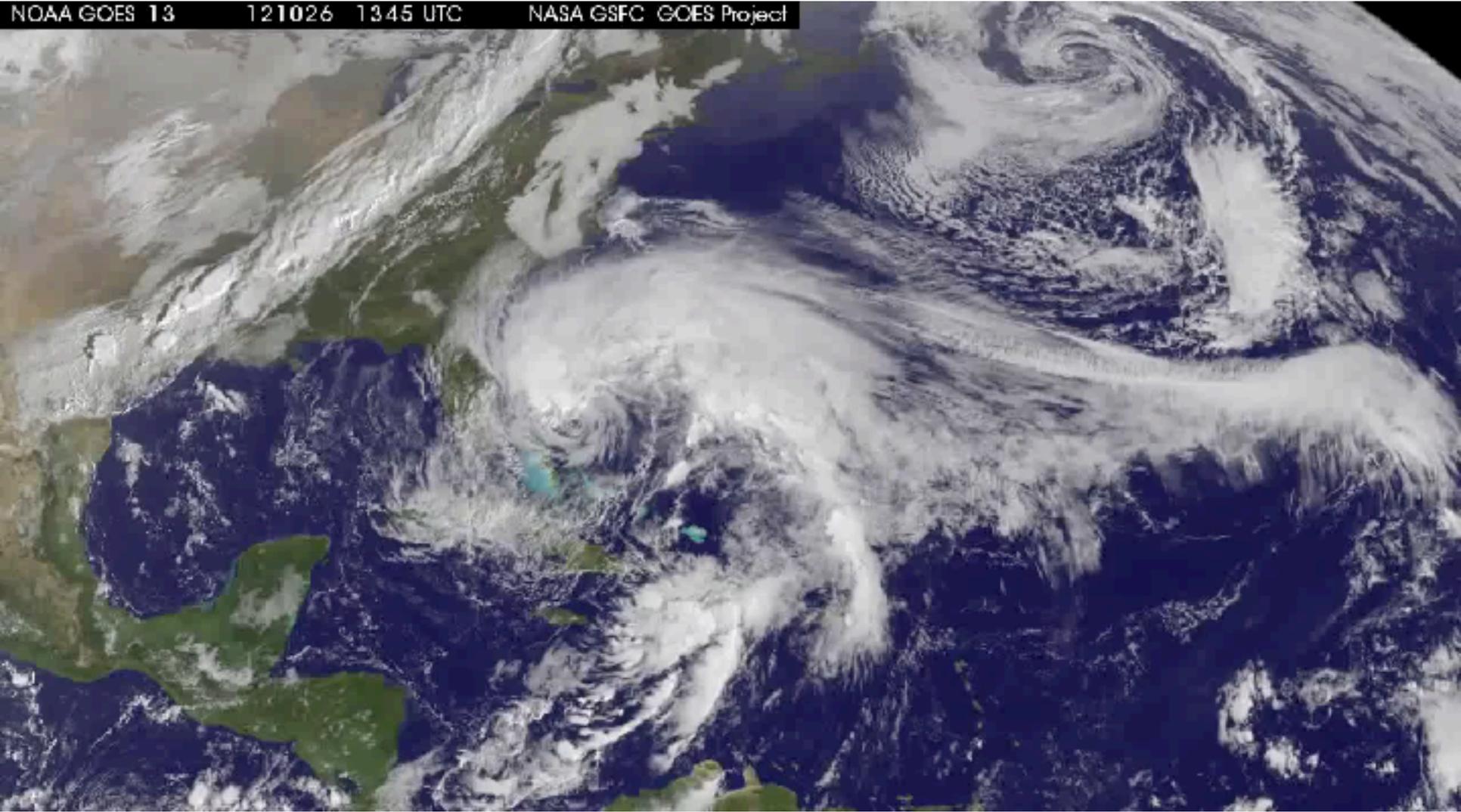
Institute für Geophysik, Astrophysik und  
Meteorologie

University of Graz, Austria

Abastumani Astrophysical Observatory at Ilia state  
University, Georgia

# Hurricane Sandy

NOAA GOES 13 121026 1345 UTC NASA GSFC GOES Project



Courtesy to NASA GSFC



Carl-Gustaf Arvid Rossby

Born: December 28, 1898  
Stockholm, Sweden

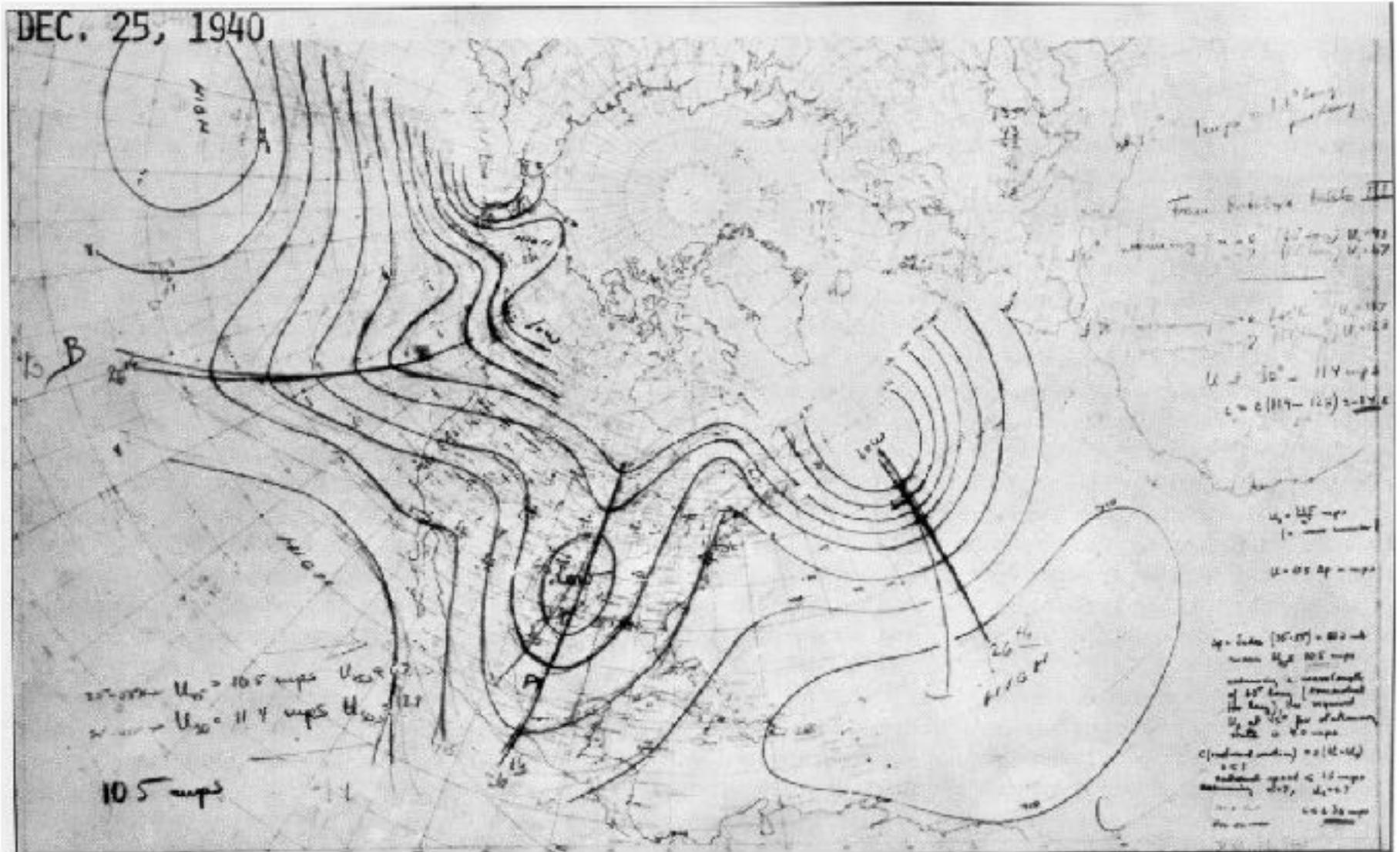
Massachusetts Institute  
of Technology

University of Chicago

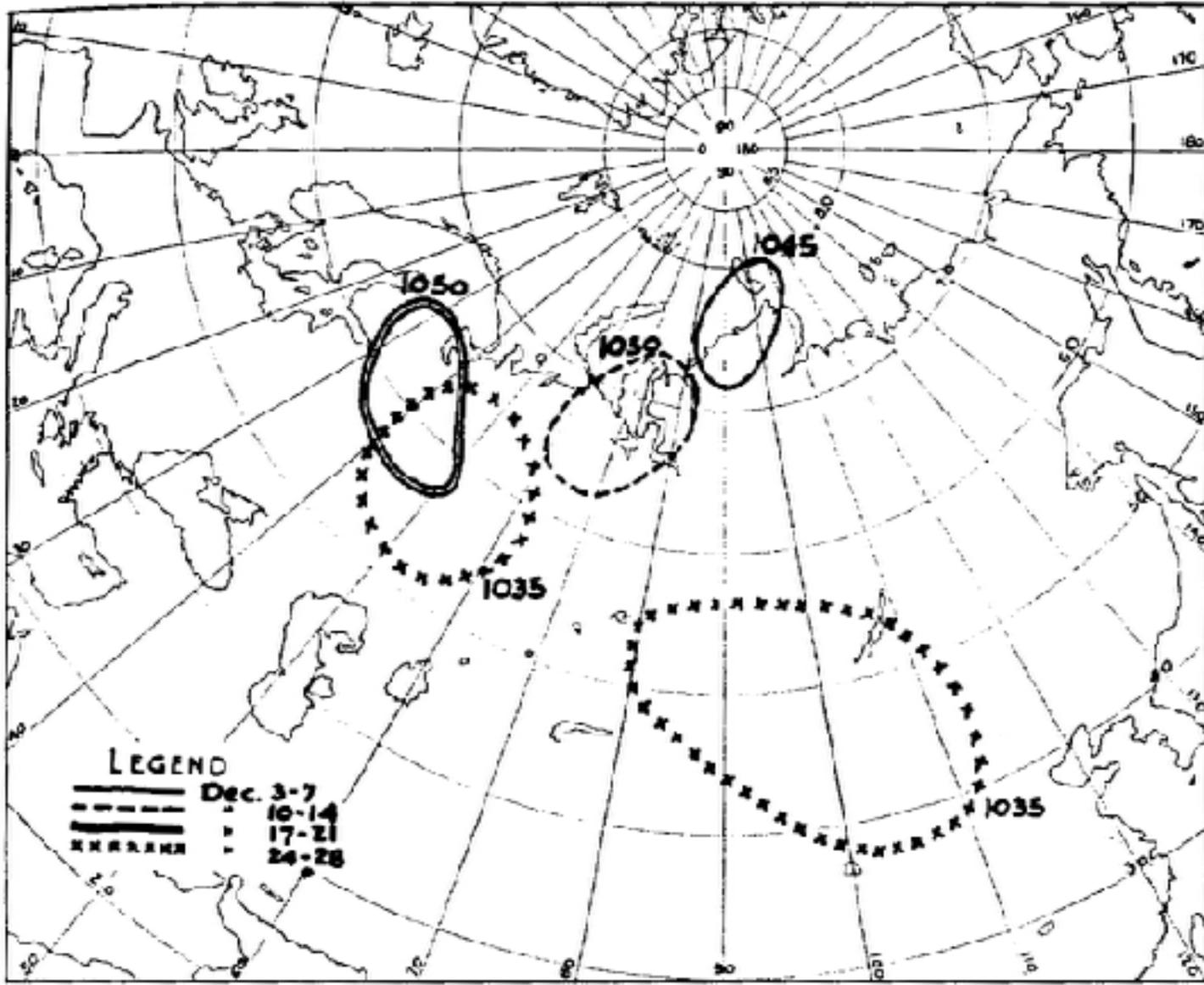
Woods Hole Oceanographic  
Institution

Swedish Meteorological and

# Rossby Waves – as seen by Rossby

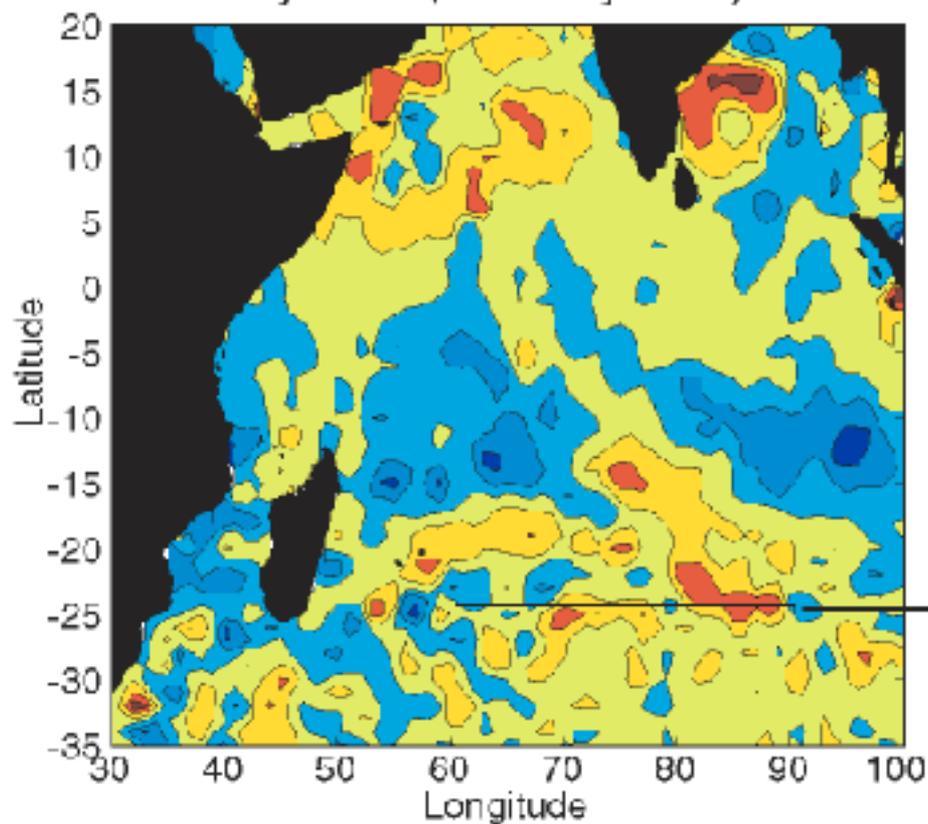


Platzman 1968

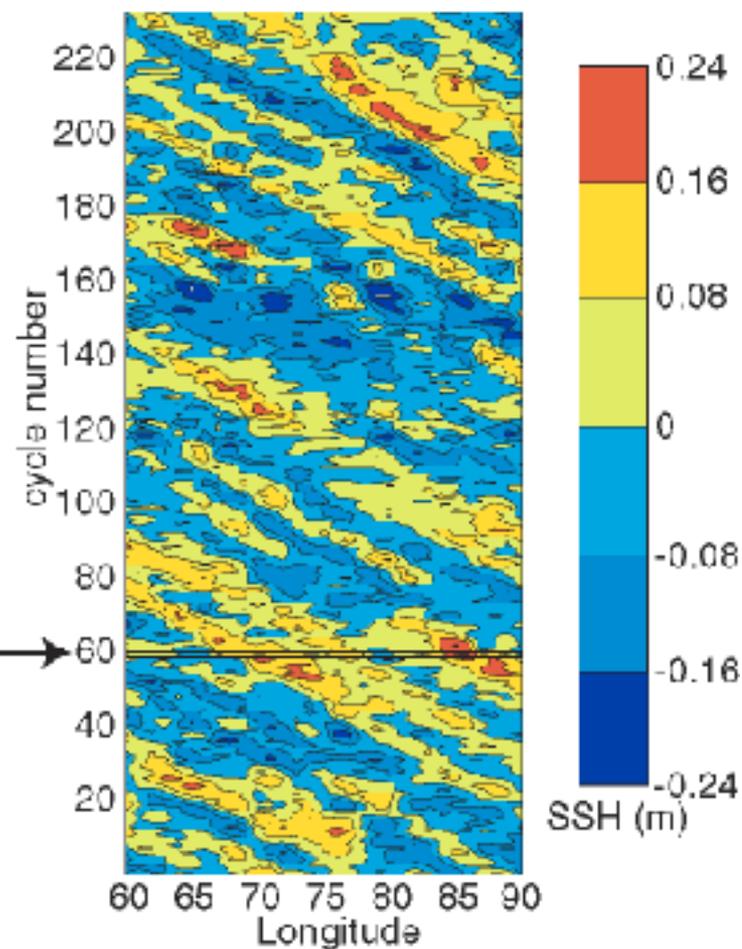


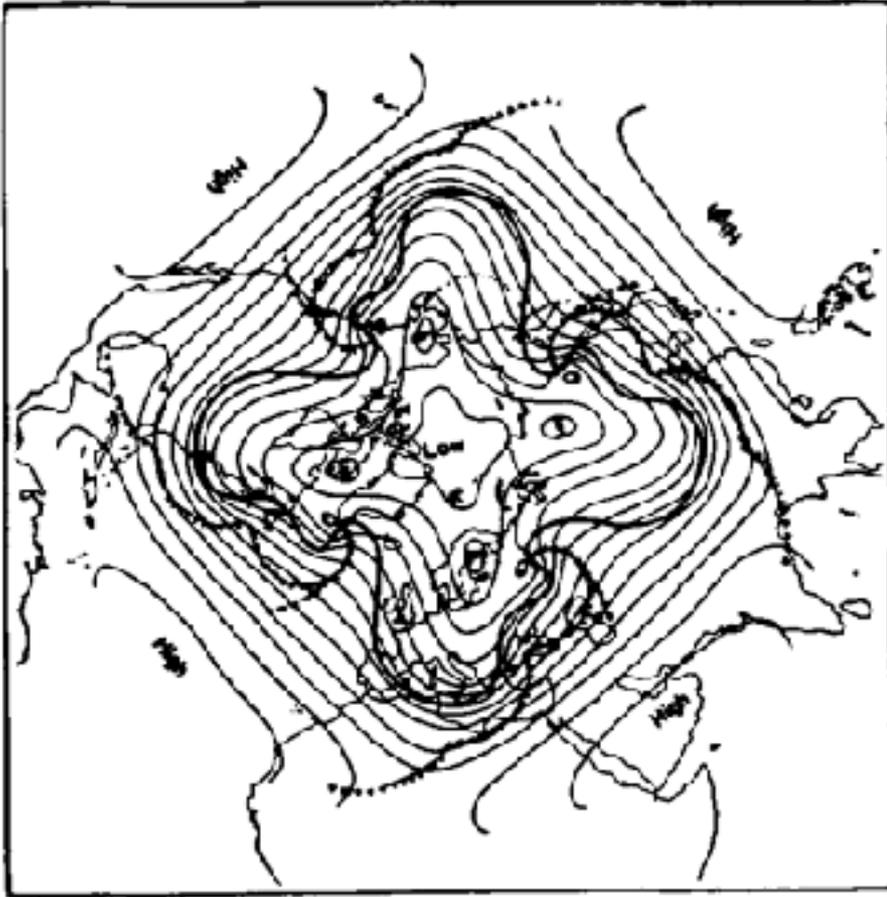
Rossby and collaborators, 1939

Sea Surface Height from TOPEX/POSEIDON  
cycle 60 (1-11 May 1994)

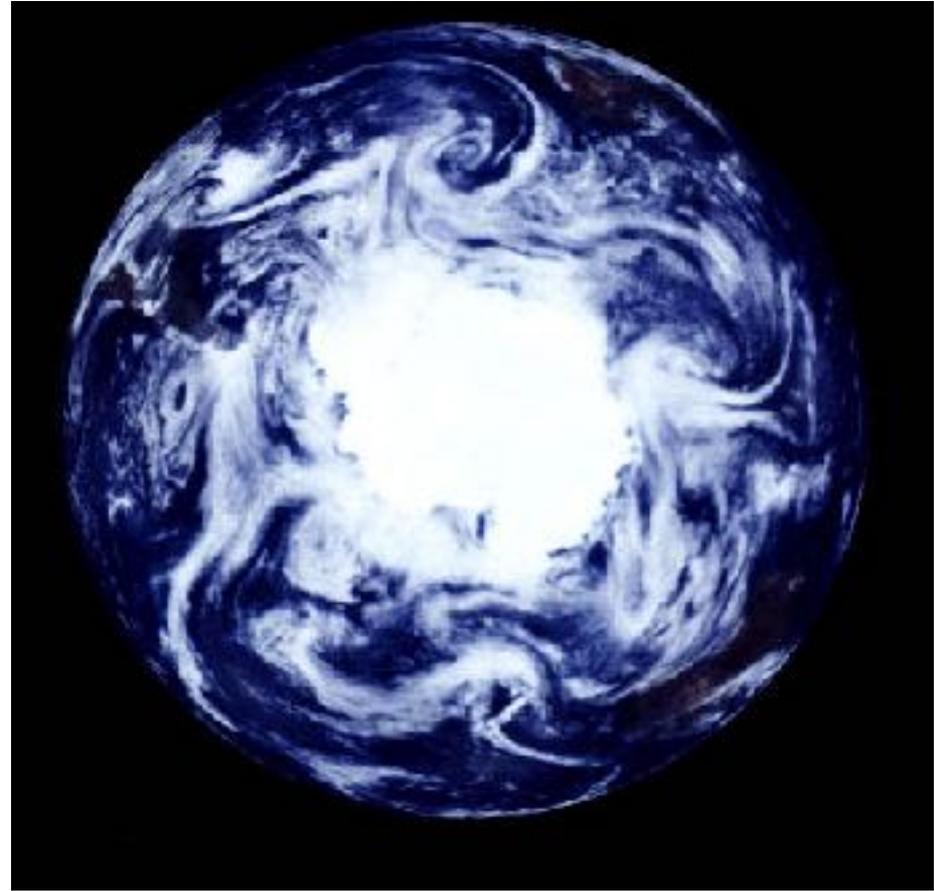


Hovmöller diagram at 25°S





Palmen 1949



The picture is taken from above the South Pole, shows a number of mid latitude cyclones circling Antarctica.

- Rossby waves may play a significant role in large-scale dynamics of Earth (and planetary) core, astrophysical discs, solar/stellar atmospheres/interiors, etc.
- Solar/stellar atmospheres/interiors and planetary cores contain **magnetic fields**.
- Therefore, the hydrodynamic Rossby wave theory should be modified in the presence of large-scale magnetic fields.

# Vorticity and magnetic field

A dynamic variable of preeminent importance in rotating fluid dynamics is **vorticity**

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

The vorticity vector is **nondivergent**

$$\nabla \cdot \boldsymbol{\omega} = 0.$$

For a fluid with uniform rotation,  $u_\varphi = \Omega r$ , the vorticity is

$$\omega_z = \frac{1}{r} \frac{\partial}{\partial r} (r u_\varphi) = \frac{1}{r} \frac{\partial}{\partial r} (r^2 \Omega) = 2\Omega.$$

Momentum equation in fluid dynamics is

$$\frac{\partial \mathbf{u}}{\partial t} - (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} + \frac{1}{c \rho} \mathbf{j} \times \mathbf{b}$$

Taking its curl gives the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nu \Delta \boldsymbol{\omega} + \nabla \times \frac{1}{mc n_e} \mathbf{j} \times \mathbf{b}$$

## Electron and proton momentum equations

$$\rho_e \frac{d\mathbf{u}_e}{dt} = -\nabla p_e - en_e \left( \mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{b} \right) - \alpha_{ei} (\mathbf{u}_e - \mathbf{u}_i),$$

$$\rho_i \frac{d\mathbf{u}_i}{dt} = -\nabla p_i + en_i \left( \mathbf{E} + \frac{1}{c} \mathbf{u}_i \times \mathbf{b} \right) + \alpha_{ei} (\mathbf{u}_e - \mathbf{u}_i)$$

Ohm's law is obtained from the electron equation

$$\mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{b} + \frac{1}{en_e} \nabla p_e = \frac{\alpha_{ei}}{e^2 n_e^2} \mathbf{j} + \frac{1}{cen_e} \mathbf{j} \times \mathbf{b}$$

$$\mathbf{j} = -en_e (\mathbf{u}_e - \mathbf{u}_i) \quad \text{current density}$$

Defining electric field as

$$\mathbf{E} = -\frac{1}{c} \mathbf{u}_i \times \mathbf{b} - \frac{1}{en_e} \nabla p_e + \frac{\alpha_{ei}}{e^2 n_e^2} \mathbf{j} + \frac{1}{cen_e} \mathbf{j} \times \mathbf{b}$$

and substituting into Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{b}}{\partial t}$$

one gets the induction equation

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \frac{mc}{e} \nabla \times \left( \frac{\nabla p}{\rho} \right) + \eta \Delta \mathbf{b} - \nabla \times \frac{1}{en_e} \mathbf{j} \times \mathbf{b}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} - \boldsymbol{\omega} \nabla \cdot \mathbf{u} - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nu \Delta \boldsymbol{\omega} + \nabla \times \frac{1}{mc n_e} \mathbf{j} \times \mathbf{b}$$

The vorticity equation is analogous to the induction equation

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u} - \mathbf{b} \nabla \cdot \mathbf{u} + \frac{mc}{e} \nabla \times \left( \frac{\nabla p}{\rho} \right) + \eta \Delta \mathbf{b} - \nabla \times \frac{1}{en_e} \mathbf{j} \times \mathbf{b}$$

The term proportional to

$$\nabla \times \left( \frac{\nabla p}{\rho} \right) = -\frac{1}{\rho^2} (\nabla \rho \times \nabla p)$$

is called the *baroclinic* term in the vorticity equation and *Biermann battery* term in the induction equation.

If the fluid density is constant, or if the density is a function of only a pressure, then the *baroclinic* term vanishes

$$\frac{1}{\rho^2} (\nabla \rho \times \nabla p) = \frac{1}{\rho^2} (\nabla \rho \times \nabla \rho) \frac{dp}{d\rho} = 0$$

and the vorticity and the induction equations become neglecting Lorentz force and Hall term

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} - \boldsymbol{\omega} \nabla \cdot \mathbf{u} + \nu \Delta \boldsymbol{\omega},$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u} - \mathbf{b} \nabla \cdot \mathbf{u} + \eta \Delta \mathbf{b}.$$

Using the continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

the vorticity and the induction equations in the ideal fluid become

$$\frac{D}{Dt} \left( \frac{\boldsymbol{\omega}}{\rho} \right) = \left( \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \right) \mathbf{u},$$

$$\frac{D}{Dt} \left( \frac{\mathbf{b}}{\rho} \right) = \left( \frac{\mathbf{b}}{\rho} \cdot \nabla \right) \mathbf{u}.$$

The vorticity equation easily leads to the statement known as Ertel theorem: *If  $\lambda$  is some conserved scalar quantity, then the potential vorticity*

$$\Pi = \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \lambda$$

*is conserved by each fluid element.*

The same theorem is valid for the magnetic field as

$$\Pi_m = \frac{\mathbf{b}}{\rho} \cdot \nabla \lambda$$

*is conserved by each fluid element.*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} - \boldsymbol{\omega} \nabla \cdot \mathbf{u} - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nu \Delta \boldsymbol{\omega} + \nabla \times \frac{1}{mc n_e} \mathbf{j} \times \mathbf{b}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u} - \mathbf{b} \nabla \cdot \mathbf{u} + \frac{mc}{e} \nabla \times \left( \frac{\nabla p}{\rho} \right) + \eta \Delta \mathbf{b} - \nabla \times \frac{1}{en_e} \mathbf{j} \times \mathbf{b}$$

Sum of vorticity and induction equations gives

$$\frac{\partial \boldsymbol{\Omega}_B}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\Omega}_B = (\boldsymbol{\Omega}_B \cdot \nabla) \mathbf{u} - \boldsymbol{\Omega}_B \nabla \cdot \mathbf{u}$$

$$\frac{d \boldsymbol{\Omega}_B}{dt} = (\boldsymbol{\Omega}_B \cdot \nabla) \mathbf{u}$$

$$\boldsymbol{\Omega}_B = \mathbf{b} + \frac{mc}{e} \boldsymbol{\omega} \quad \text{is conserved.}$$

The vorticity of the fluid as observed from an inertial, nonrotating frame is called **absolute vorticity**

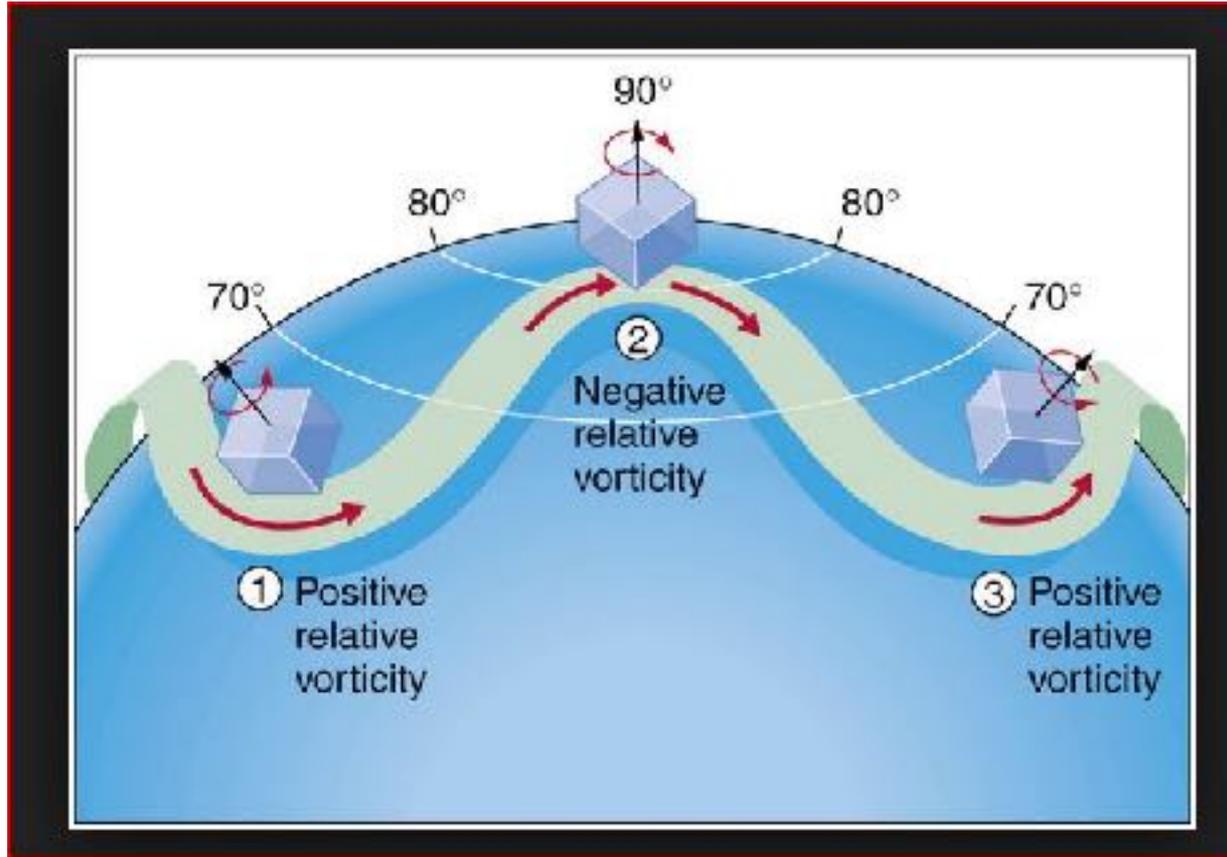
$$\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$$

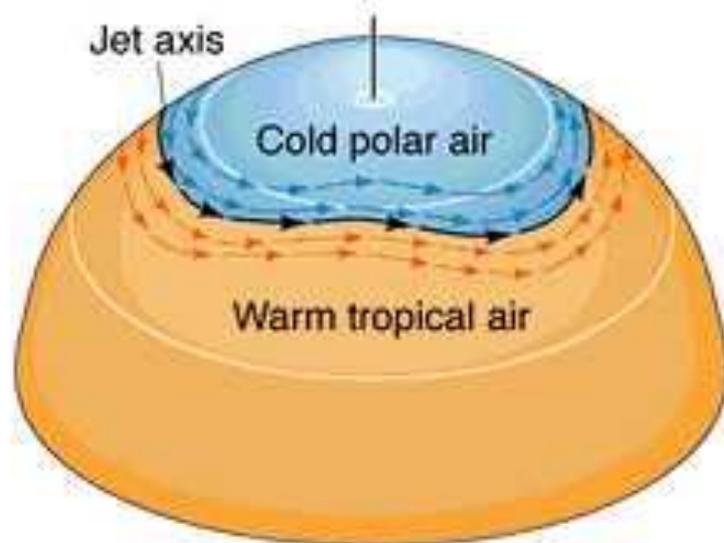
and this quantity is conserved during the fluid motion on a rotating sphere.

The vorticity of rotating sphere (e.g. Earth) is maximal at poles and tends to zero at the equator.

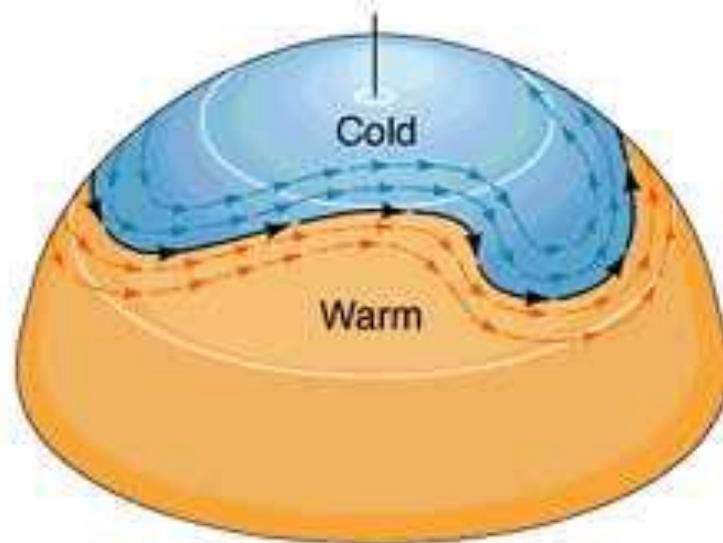
**Rossby waves** are produced from the conservation of **absolute vorticity**.

- As an air parcel moves northward or southward over different latitudes, it experiences change in **Earth vorticity**.
- In order to conserve the **absolute vorticity**, the air has **to rotate** to produce relative vorticity.
- The **rotation** due to the relative vorticity bring the air **back** to where it was.

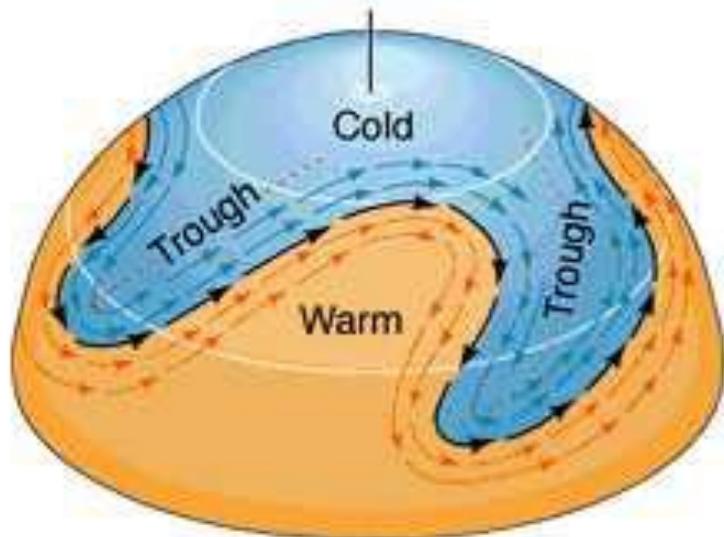




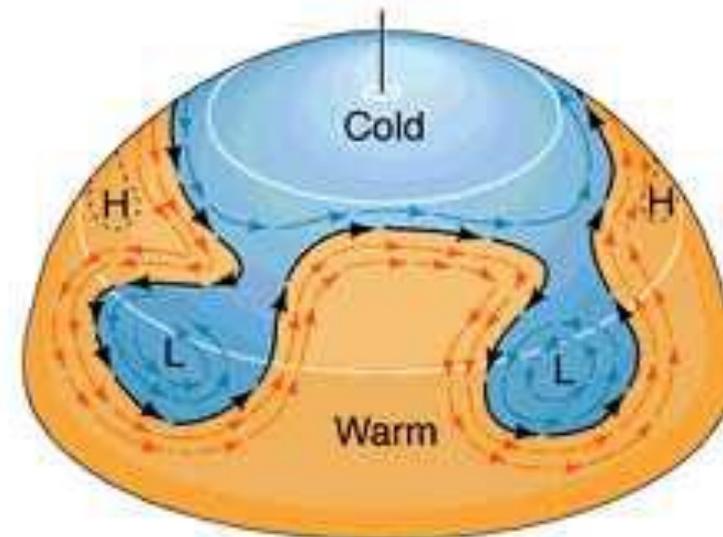
The jet stream begins to undulate...



Rossby waves begin to form.



Waves are strongly developed. The cold air occupies troughs of low pressure.



When the waves are pinched off, they form cyclones of cold air.

# Rossby waves (hydrodynamics)

Momentum equation in rotating frame (with angular velocity  $\boldsymbol{\Omega}$ )

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - 2\boldsymbol{\Omega} \times \mathbf{u} + \mathbf{g},$$

where  $-2\rho\boldsymbol{\Omega} \times \mathbf{u}$  is the Coriolis force.

Ratio of convective and Coriolis terms is called a Rossby number

$$Ro = \frac{U}{L\Omega}$$

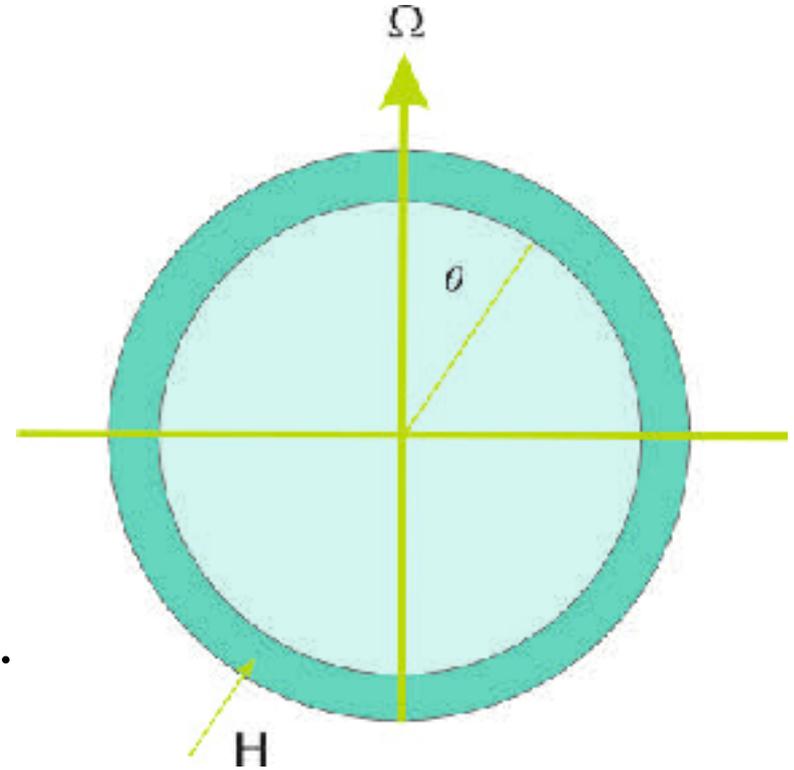
When  $Ro \ll 1$  then the rotation effects are significant.

In 18<sup>th</sup> century, Laplace formulated his “tidal” equations

$$\frac{\partial u_\varphi}{\partial t} + 2\Omega \cos\theta u_\theta = -\frac{g}{R \sin\theta} \frac{\partial h}{\partial \varphi},$$

$$\frac{\partial u_\theta}{\partial t} - 2\Omega \cos\theta u_\varphi = -\frac{g}{R} \frac{\partial h}{\partial \theta},$$

$$\frac{\partial h}{\partial t} + \frac{H}{R \sin\theta} \left[ \frac{\partial}{\partial \theta} (\sin\theta u_\theta) + \frac{\partial u_\varphi}{\partial \varphi} \right] = 0.$$



$\theta$  is co-latitude,  $\varphi$  is longitude,  $g$  is the acceleration,  $H$  is the layer thickness,  $h$  is the surface elevation,  $\Omega$  is the angular frequency.

Rectangular coordinates:

$$\frac{\partial u_x}{\partial t} - fu_y = -g \frac{\partial h}{\partial x},$$

$$\frac{\partial u_y}{\partial t} + fu_x = -g \frac{\partial h}{\partial y},$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (Hu_x) + \frac{\partial}{\partial y} (Hu_y) = 0.$$

$f = 2\Omega \sin \vartheta$  is the Coriolis parameter.

$\vartheta = 90^\circ - \theta$  is the latitude.

These equations can be cast into one equation

$$\frac{\partial}{\partial t} \left[ \frac{1}{c^2} \left( \frac{\partial^2}{\partial t^2} + f^2 \right) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] u_y - \frac{\partial f}{\partial y} \frac{\partial u_y}{\partial x} = 0.$$

$c = \sqrt{gH}$  is the surface gravity speed.

If one neglects the surface elevation  $h \approx 0$  or  $\frac{h}{H} \ll 1$

$$\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_y + \frac{\partial f}{\partial y} \frac{\partial u_y}{\partial x} = 0.$$

**This approximation eliminates surface gravity (or Poincare) waves and induces small change in Rossby wave dispersion relation.**

At this point came up Rossby with his  $\beta$  - **plane approximation**

When spatial scales of considered process is less than sphere radius then one can expand the Coriolis parameter at a given latitude as

$$f = f_0 + \beta y,$$

$$\beta = \frac{\partial f}{\partial y} = \frac{2\Omega}{R} \cos\vartheta = \text{const.}$$

$$\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_y + \beta \frac{\partial u_y}{\partial x} = 0.$$

Fourier analysis of the form  $\exp(-i\tilde{\omega}t + ik_x x + ik_y y)$  leads to the dispersion relation of Rossby (or planetary) waves

$$\tilde{\omega} = -\frac{\beta k_x}{k_x^2 + k_y^2}.$$

Rossby waves always propagate in the opposite direction of rotation!

For purely toroidal propagation  $\tilde{\omega} = -\frac{\beta}{k_x}$ .

Phase speed  $v_{ph} = \frac{\tilde{\omega}}{k_x} = -\frac{\beta}{k_x^2 + k_y^2}$ .

Long wavelength waves propagate faster!

Group speed  $\mathbf{v}_g = \left( \frac{\partial \tilde{\omega}}{\partial k_x}, \frac{\partial \tilde{\omega}}{\partial k_y} \right) = \left( -\beta \frac{k_y^2 - k_x^2}{(k_x^2 + k_y^2)^2}, \beta \frac{2k_x k_y}{(k_x^2 + k_y^2)^2} \right)$

If there is a constant zonal flow then the phase speed can be written as (Rossby 1939)

$$c = U - \frac{\beta L^2}{4\pi^2},$$

It appears that the waves become stationary when

$$c = U - \frac{\beta L_s^2}{4\pi^2} = 0, \quad L_s = 2\pi \sqrt{\frac{U}{\beta}}.$$

$$c = U \left( 1 - \frac{L^2}{L_s^2} \right),$$

Long wavelength waves propagate westward and short wavelength waves propagate eastward!

$\beta=1.6 \cdot 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$  at mid-latitudes.

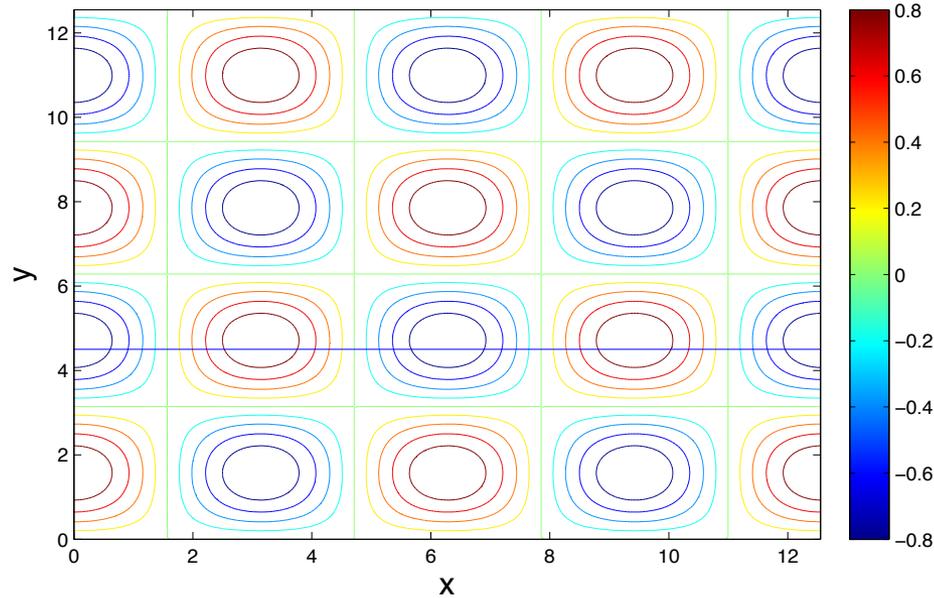
For the wavelength of 10000 km, one gets the period of 5.6 days.

Phase speed - 20 m/s.

The observed Rossby wave period on the Earth is 4-6 days  
(Yanai and Maruyama 1966, Wallace 1973, Madden 1979).

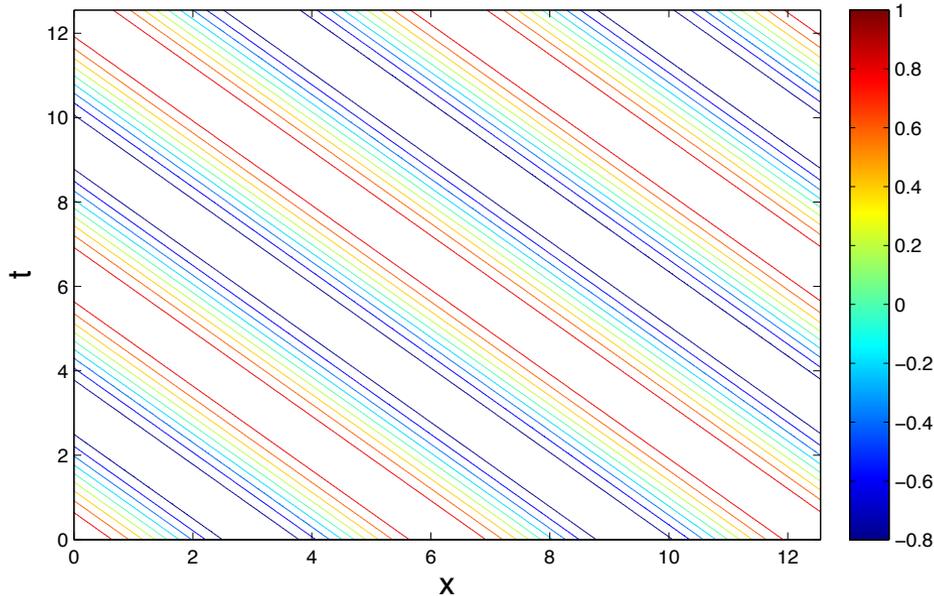
The ratio of Rossby wave and Earth rotation periods is around 6!

Rossby wave,  $t=0$



The red corresponds to high pressure regions and the blue to low. The lower panel shows a “Hovmuller” diagram of the phases at  $y = 4.5$  as a function of time.

Hovmuller diagram,  $y=4.5$



Courtesy to Lacasce

When the spatial scale of considered process is longer than the sphere radius then one should use the spherical coordinates

$$\frac{\partial u_\theta}{\partial t} - 2\Omega \cos\theta u_\varphi = -\frac{g}{R} \frac{\partial p}{\partial \theta},$$

$$\frac{\partial u_\varphi}{\partial t} + 2\Omega \cos\theta u_\theta = -\frac{1}{\rho R \sin\theta} \frac{\partial p}{\partial \varphi},$$

$$\frac{\partial}{\partial \theta} (\sin\theta u_\theta) + \frac{\partial u_\varphi}{\partial \varphi} = 0.$$

Fourier analysis with  $\exp(-i\tilde{\omega}t + im\varphi)$  leads to the equation

$$\left[ \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} - \frac{m^2}{1 - \mu^2} - \frac{2m\Omega}{\tilde{\omega}} \right] \tilde{u}_\theta = 0,$$

where  $\mu = \cos\theta$ ,  $m$  is the toroidal wave number and  $\tilde{u}_\theta = \sin\theta u_\theta$ .

If 
$$-\frac{2m\Omega}{\tilde{\omega}} = n(n+1)$$

then the equation is associated Legendre differential equation, those typical solutions are associated Legendre polynomials

$$\tilde{u}_\theta = P_n^m(\cos\theta),$$

where  $n-m$  is a number of nodes along the latitude.

It defines the dispersion relation for spherical Rossby waves (Haurwitz 1940, Longuet-Higgins 1968, Papaloizou and Pringle 1978)

$$\tilde{\omega} = -\frac{2m\Omega}{n(n+1)}.$$
$$v_{ph} = \frac{\tilde{\omega}}{m} = -\frac{2\Omega}{n(n+1)}.$$

# Rossby waves (magnetohydrodynamics)

First consideration of magnetic field effects on Rossby waves was done by Hide (1966).

He considered incompressible 2D MHD approximation with uniform 2D magnetic field.

Then Gilman (2000) wrote MHD shallow water equations for nearly horizontal magnetic field

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B} - \mathbf{g} \nabla H + f \mathbf{u} \times \mathbf{z}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u}$$

$$\frac{\partial H}{\partial t} = -\nabla \cdot H \mathbf{u}$$

Here  $\mathbf{B}$  and  $\mathbf{u}$  are horizontal magnetic field and velocity,  $H$  is the thickness of the layer,  $\mathbf{g}$  is the reduced gravity.

The **divergence-free condition** for magnetic fields is now written as

$$\nabla \cdot (\mathbf{HB}) = 0$$

This states simply that at every point the **magnetic flux** associated with the **horizontal magnetic field**, which are independent with height, is **conserved**.

The **total magnetic field** is made up of **horizontal fields** independent of the vertical together with a **small vertical field** that is, like the vertical velocity, a linear function of height, being zero at the bottom and maximum at the top.

## Linear equations in x-y plane

$$\frac{\partial u_x}{\partial t} - fu_y = \frac{B_x}{4\pi\rho} \frac{\partial b_x}{\partial x} - g \frac{\partial h}{\partial x}$$

$$\frac{\partial u_y}{\partial t} + fu_x = \frac{B_x}{4\pi\rho} \frac{\partial b_y}{\partial x} - g \frac{\partial h}{\partial y}$$

$$\frac{\partial b_x}{\partial t} = B_x \frac{\partial u_x}{\partial x}$$

$$\frac{\partial b_y}{\partial t} = B_x \frac{\partial u_y}{\partial x}$$

$$\frac{\partial h}{\partial t} + H_0 \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0$$

$B_x$  is the unperturbed horizontal magnetic field.

$$\frac{\partial^2 u_y}{\partial y^2} + \left[ \frac{\omega^2}{C_0^2} - k_x^2 - \frac{k_x^2 v_A^2}{C_0^2} - \frac{\omega^2 f^2}{C_0^2 (\omega^2 - k_x^2 v_A^2)} - \frac{k_x \omega}{\omega^2 - k_x^2 v_A^2} \frac{\partial f}{\partial y} \right] u_y = 0.$$

$C_0 = \sqrt{gH_0}$  is surface gravity speed.

$v_A = \frac{B_x}{\sqrt{4\pi\rho}}$  is the Alfvén speed.

At a given latitude we can expand the Coriolis parameter as

$$f = f_0 + \beta y, \quad \beta = \frac{2\Omega_0}{R_0} \cos \Theta.$$

Away from the equator  $\beta y \ll$  then we get

$$\frac{\partial^2 u_y}{\partial y^2} + \left[ \frac{\omega^2}{C_0^2} - k_x^2 - \frac{k_x^2 v_A^2}{C_0^2} - \frac{\omega^2 f^2}{C_0^2 (\omega^2 - k_x^2 v_A^2)} - \frac{k_x \omega \beta}{\omega^2 - k_x^2 v_A^2} \right] u_y = 0.$$

This equation gives the dispersion relation (Zaqarashvili et al. 2007)

$$\omega^4 - [2k_x^2 v_A^2 + f_0^2 + C_0^2 (k_x^2 + k_y^2)] \omega^2 - C_0^2 k_x \beta \omega + k_x^2 v_A^2 [k_x^2 v_A^2 + C_0^2 (k_x^2 + k_y^2)] = 0.$$

For  $v_A \ll C_0$  this equation has two different branches

Higher frequency branch (Poincaré waves):

$$\omega^2 = f_0^2 + C_0^2 (k_x^2 + k_y^2)$$

Low frequency branch (magnetic Rossby waves):

$$\omega^2 + \frac{k_x \beta}{k_x^2 + k_y^2} \omega - v_A^2 k_x^2 = 0$$

Hide (1966) considered only x-y plane and obtained:

$$\tilde{\omega}^2 + \frac{k\beta}{k^2 + l^2} \tilde{\omega} - v_A^2 (k \cos \vartheta + l \sin \vartheta)^2 = 0.$$

The dispersion relation has two solutions: fast (high-frequency) and slow (low-frequency) magnetic Rossby waves.

High-frequency solution corresponds to fast magnetic Rossby waves.

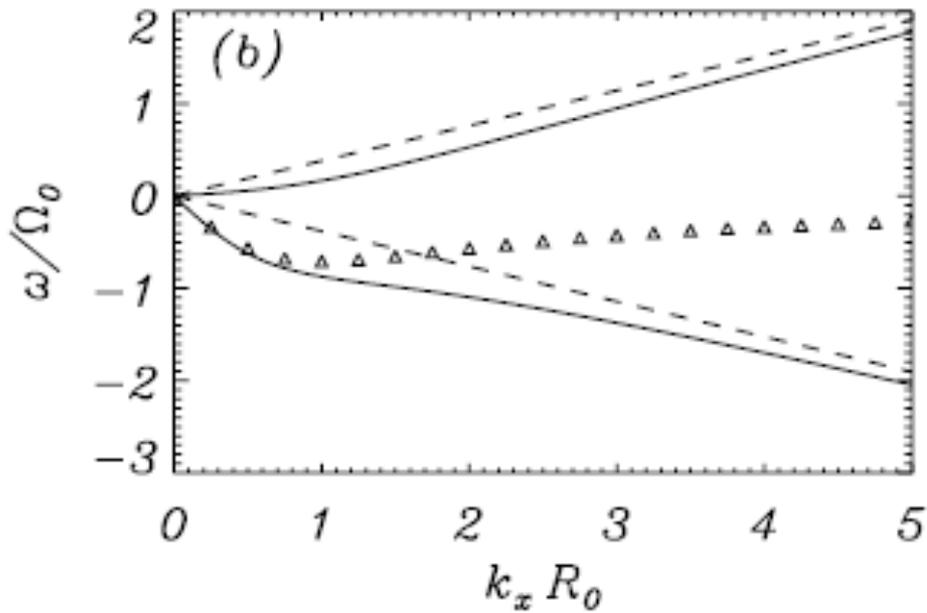
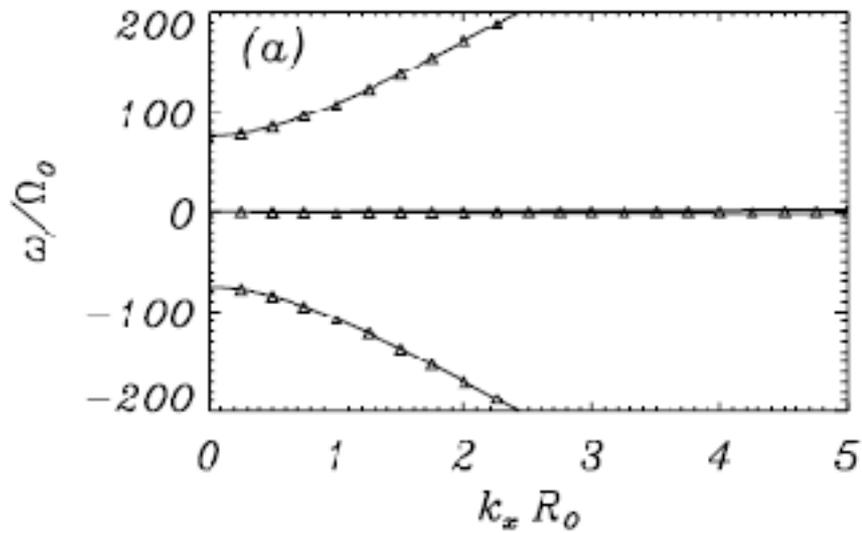
Low-frequency solution corresponds to slow magnetic Rossby waves

$$\omega \approx \frac{k_x v_A^2 (k_x^2 + k_y^2)}{\beta}.$$

*The magnetic field splits the ordinary HD Rossby waves into fast and slow magnetic Rossby modes!*

No magnetic field:  $\tilde{\omega} = -\frac{k_x \beta}{k_x^2 + k_y^2}$  HD Rossby waves

No rotation:  $\tilde{\omega}^2 = v_A^2 k_x^2$  Alfvén waves



Solid lines: fast and slow modes

Triangles: HD Rossby waves

Dashed lines: Alfvén waves

Zaqarashvili et al. 2007, A&A

## Linear MHD equations in rotating frame (spherical coordinates)

$$B = (0, B_\phi, 0), B_\phi = \sin\theta B_0$$

$$\frac{\partial u_\theta}{\partial t} - 2m\Omega_0 \cos\theta u_\phi + \frac{g}{R} \sin\theta \frac{\partial h}{\partial \theta} - \frac{B_0}{4\pi\rho R} \frac{\partial b_\phi}{\partial \varphi} + 2 \frac{B_0}{4\pi\rho R} \cos\theta b_\phi = 0$$

$$\frac{\partial u_\phi}{\partial t} - 2\Omega_0 \cos\theta u_\theta + \frac{g}{R} \sin\theta \frac{\partial h}{\partial \varphi} - \frac{B_0}{4\pi\rho R} \frac{\partial b_\phi}{\partial \varphi} - 2 \frac{B_0}{4\pi\rho R} \cos\theta b_\theta = 0$$

$$\sin^2\theta \frac{\partial h}{\partial t} + \frac{H_0}{R} \sin\theta \frac{\partial u_\theta}{\partial \theta} + \frac{H_0}{R} \frac{\partial u_\phi}{\partial \varphi} = 0$$

$$\frac{\partial b_\theta}{\partial t} - \frac{B_0}{R} \frac{\partial u_\phi}{\partial \varphi} = 0$$

$$\frac{\partial b_\phi}{\partial t} + \frac{B_0}{R} \sin\theta \frac{\partial u_\theta}{\partial \theta} = 0$$

$H_0$  is the thickness of the layer.

For  $h \rightarrow 0$  Fourier analysis with  $\exp(-i\tilde{\omega}t + im\varphi)$  leads to

$$\left[ \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} - \frac{m^2}{1 - \mu^2} - \frac{2m\Omega R^2 \tilde{\omega} + 2m^2 v_A^2}{R^2 \tilde{\omega}^2 - m^2 v_A^2} \right] \tilde{u}_\theta = 0.$$

where  $\mu = \cos\theta$  and  $m$  is the toroidal wave number.

$$\text{If } \frac{2m\Omega_0 R^2 \omega + 2m^2 V_A^2}{R^2 \omega^2 - m^2 V_A^2} = n(n+1)$$

then the equation is associated Legendre differential equation, those typical solutions are associated Legendre polynomials

$$\tilde{u}_\vartheta = P_n^m(\cos\vartheta)$$

It defines the dispersion relation for spherical magnetic Rossby waves  
(Zaqarashvili et al. 2007)

$$\left(\frac{\omega}{\Omega_0}\right)^2 + \frac{2m}{n(n+1)}\frac{\omega}{\Omega_0} + \frac{B_0^2 m^2}{\mu\rho\Omega_0^2 R^2} \frac{2-n(n+1)}{n(n+1)} = 0.$$

The magnetic field causes the splitting of ordinary HD mode into the fast and slow magnetic Rossby waves.

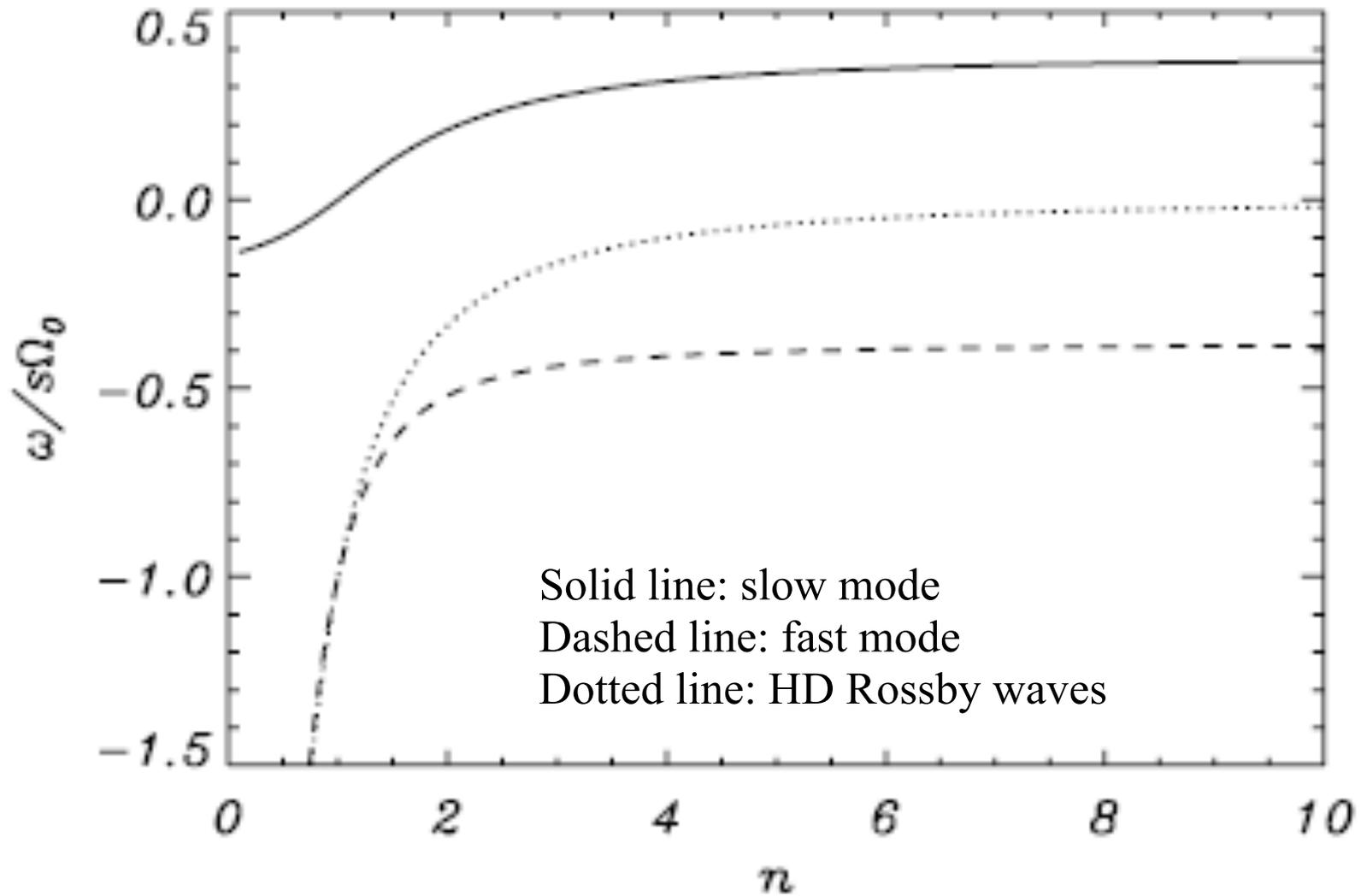
In nonmagnetic case it transforms into HD Rossby wave solution

$$\omega = -\frac{2m\Omega_0}{n(n+1)}.$$

For slow magnetic Rossby waves

$$\omega = -m\Omega_0 \frac{B_0^2}{\mu\rho\Omega_0^2 R^2} \frac{2-n(n+1)}{2}.$$

Period of particular harmonics depend on the magnetic field strength.



Zaqarashvili et al. 2007

For  $h \neq 0$  Fourier analysis with  $\exp(-i\tilde{\omega}t + im\varphi)$  leads to the complicated second order equation, which for the magnetic field profile  $B = (0, B_\phi, 0)$ ,  $B_\phi = \sin\theta \cos\theta B_0$  was solved analytically.

$$1) \quad \varepsilon = \frac{4\Omega^2 R^2}{gH_0} \ll 1 \quad (\text{strongly stable stratification})$$

The solution is presented in terms of spheroidal wave functions

$$\tilde{u}_\vartheta = S_{nm}(\varepsilon, \cos\vartheta)$$

and the dispersion relation of magnetic Rossby waves is  
(Zaqarashvili et al. 2009)

$$\left(\frac{\omega}{\Omega_0}\right)^2 + \frac{2m}{n(n+1)} \frac{\omega}{\Omega_0} + \frac{B_0^2 m^2}{4\pi\rho\Omega_0^2 R^2} \frac{1}{n(n+1)} = 0.$$

Hence, the dispersion relation of magnetic Rossby waves depends on the magnetic field structure.

$$1) \quad \varepsilon = \frac{4\Omega^2 R^2}{gH_0} \gg 1 \quad (\text{weakly stable stratification})$$

In this case the governing equation is transformed into Weber equation, which has the solution in terms of Hermite polynomials.

The solution is concentrated near the equator and hence it describes equatorially trapped waves.

The dispersion relation for magnetic Rossby waves is (Zaqarashvili et al. 2009)

$$\left( \frac{\omega}{\Omega_0} \right)^2 + \frac{2m}{(2\nu + 1)\sqrt{\varepsilon}} \frac{\omega}{\Omega_0} + \frac{B_0^2 m^2}{4\pi\rho\Omega_0^2 R^2} \frac{1}{(2\nu + 1)\sqrt{\varepsilon}} = 0.$$

# Final remarks

- Rossby waves arise due to the conservation of absolute vorticity and govern the large scale dynamics on rotating spheres.
- Horizontal magnetic field splits HD Rossby waves into fast and slow modes.
- The magnetic field and differential rotation may lead to the instability of magnetic Rossby waves in solar/stellar interiors and in astrophysical discs.
- Rossby waves can be important to explain solar/stellar activity variations.
- Observed and theoretical periods can be used to probe the dynamo layers of the Sun and solar-like stars.