Branching Processes and SOC

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Basic Idea

Each individual has a random number of children following a given statistical distribution.

Simple Example

Given maximum number of children n_b; each of them occurs with a given probability p.

$$\Downarrow$$

Number of children follows a binomial distribution with an expected value $\overline{n}_c = n_b\, p$.

Criticality

Subcritical (extinction) for $\overline{n}_c < 1$

Supercritical (population explodes) for $\overline{n}_c > 1$

Critical point (tuned!) for $\overline{n}_c = 1$

$$\Downarrow$$

Power-law distribution of the number of descendants with a scaling exponent

$$\tau = \frac{3}{2}$$

(at least for the simple example)

Differences towards the most Widespread SOC Models

- SOC models carry local information.
- "Avalanching" is random in branching processes.

	Driving	Triggering	Avalanching
BTW model	random	random	deterministic
OFC model	deterministic	deterministic	deterministic
Forest-fire model	random	random	deterministic
Branching	–	–	random

Branching with Local Probability

Spirit

- Attempt to design a simple SOC model by extending the concept of branching.
- First paper: S. Hergarten (2012) Branching with local probability as a paradigm of self-organized criticality. Phys. Rev. Lett., 109: 148001

Branching with Local Probability

First Set of Model Rules

Setup: Consider n sites and assign a variable p_i (probability) to each site.

Driving and triggering: Select a site i randomly and increase p_i by a given amount. The site relaxes with the probabilty p_i.

Avalanching: Each active site triggers a given number n_b of randomly selected further sites ("children"):

- Transfer an amount $\frac{p_i}{n_b}$ to each of these sites and set $p_i = 0$ (conservative relaxation):

$$
\begin{aligned}
p_j &:= p_j + \frac{p_i}{n_b} \quad \text{for all "children" } j \text{ of } i \\
p_i &:= 0
\end{aligned}
$$

- Each "child" j relaxes with the probability p_j.

Branching with Local Probability

Boundary Conditions

Static: Define a set of sites which cannot participate in an avalanche.

Dynamic: Assume that each site can only be triggered once during an avalanche; set $p_i = 0$ at the second trigger.

Result

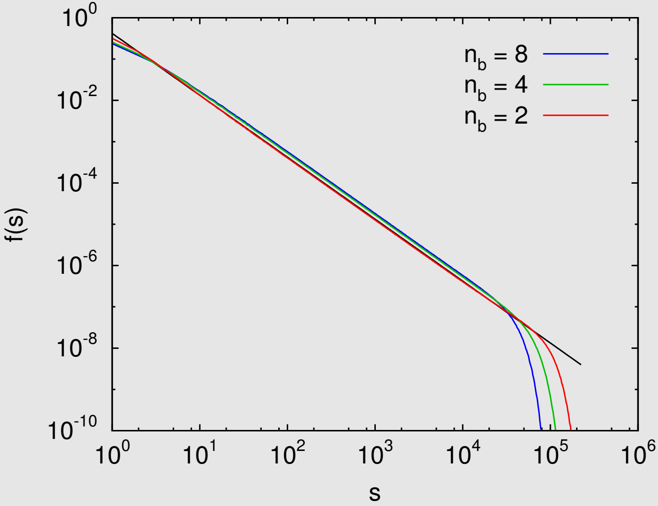
Power-law distribution of the avalanches with $\tau = \frac{3}{2}$, stable against

- the size of the increment used for driving and
- the number of "children" n_b.

The statistical distribution of the probabilities p_i self-organizes to maintain the power-law distribution.

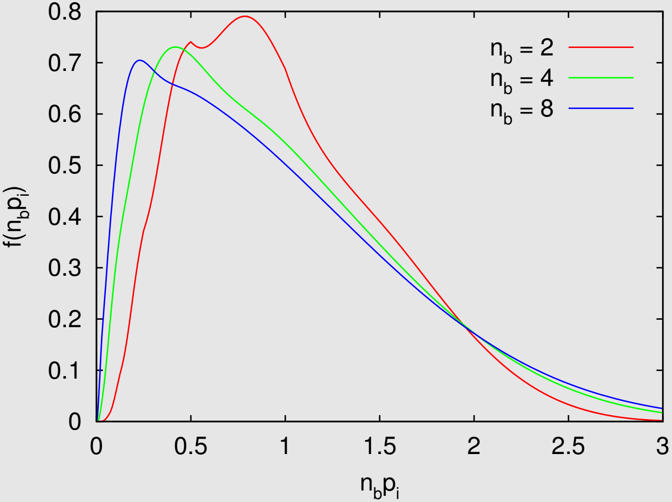
Branching with Local Probability

Avalanche-Size Distribution

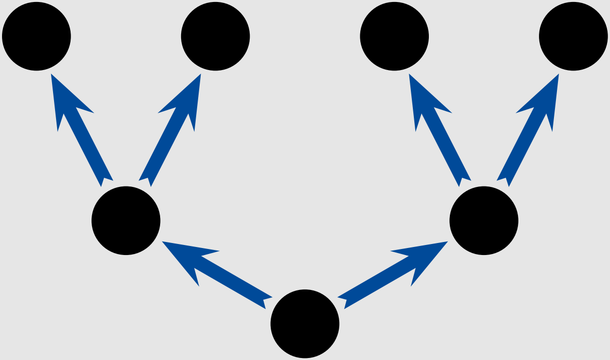


Branching with Local Probability

Distribution of the Probabilities



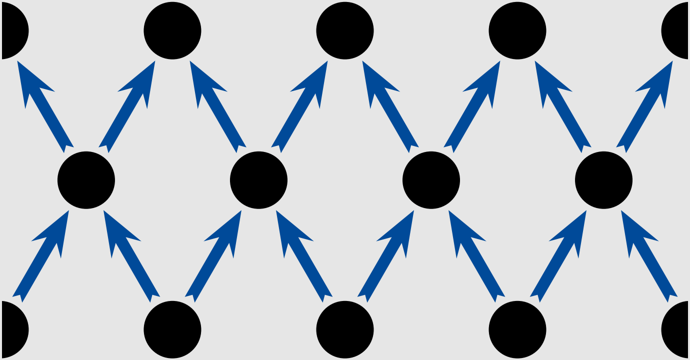
Binary Tree



- Seems to work with the same scaling exponent $\tau = \frac{3}{2}$.
- Difficult to test due to high numerical effort.

Generalization to Trees, Lattices or other Graphs

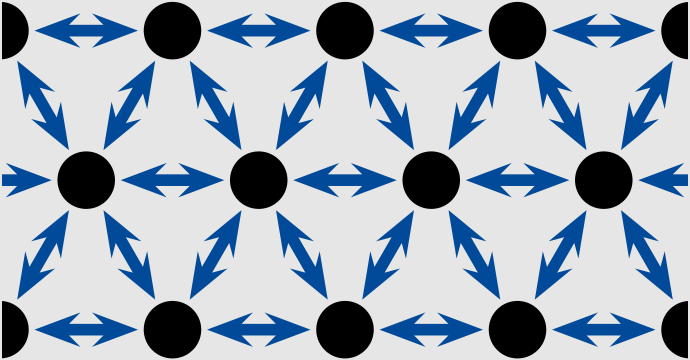
Directed Lattice



- Less clearer power-law distribution with a tendency $\tau < \frac{3}{2}$.
- Better for larger branching numbers $n_b > 2$?

Generalization to Trees, Lattices or other Graphs

Lattice



Not yet tested.

Further Generalization

Probability Function

Use an arbitrary monotonic function $f(p_i)$ as the probability instead of p_i.

$$\Downarrow$$

Seems to work, at least for the random-neighbor version.

Conclusions and Open Questions

Conclusions

- Simple model with a robust power-law distribution and a universal scaling exponent $\tau = \frac{3}{2}$.
- Conservation seems to be crucial for criticality.
- Several modifications where the model seems to be almost SOC.

Open Questions

- Temporal correlations?
- Relationship to real-world phenomena or other models, e. g., gradient-based sandpile / rockfall model?
- Analytical solution for the simplest case, perhaps random neighbor and $n_b \to \infty$?
- ...