Self-Organised near to Criticality (= scale invariance)

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and

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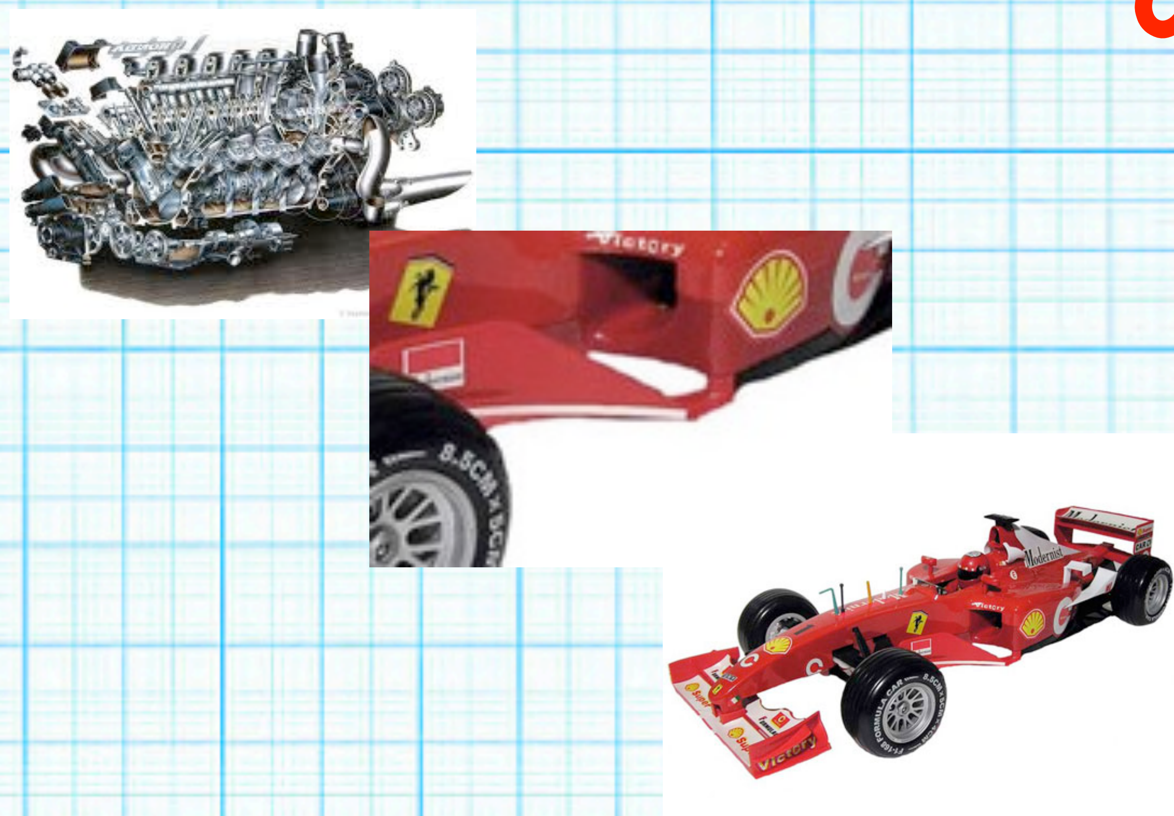
From: http://ethiopia.limbo13.com/index.php/cornrows_and_fractals/

Outline:

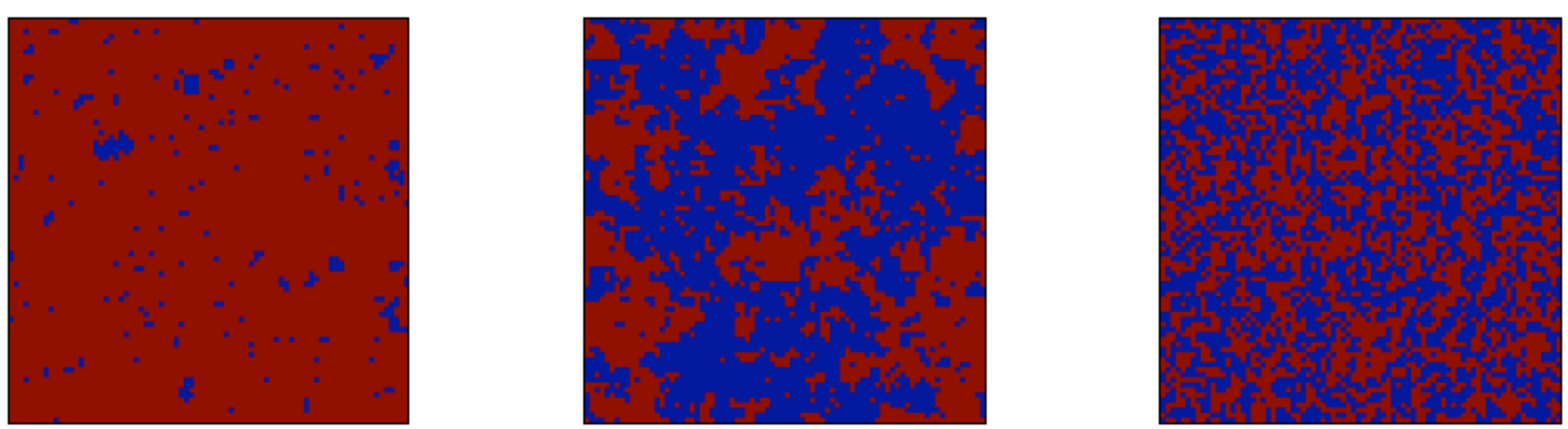
- Criticality and self-similarity or scale invariance
- Macroscopic spatio-temporal scale invariance in the brain
- SOC or avalancheology
- 3/2
- The rain and brain approach - Dantology

Scale



or no scale

Criticality and self-similarity

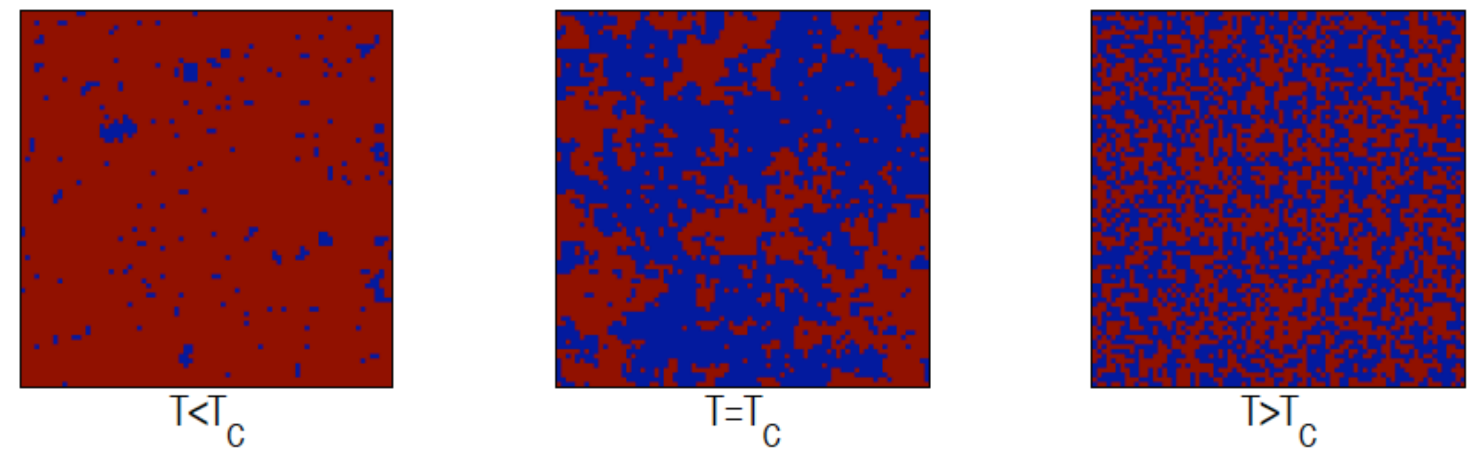


$T<T_C$ $T=T_C$ $T>T_C$

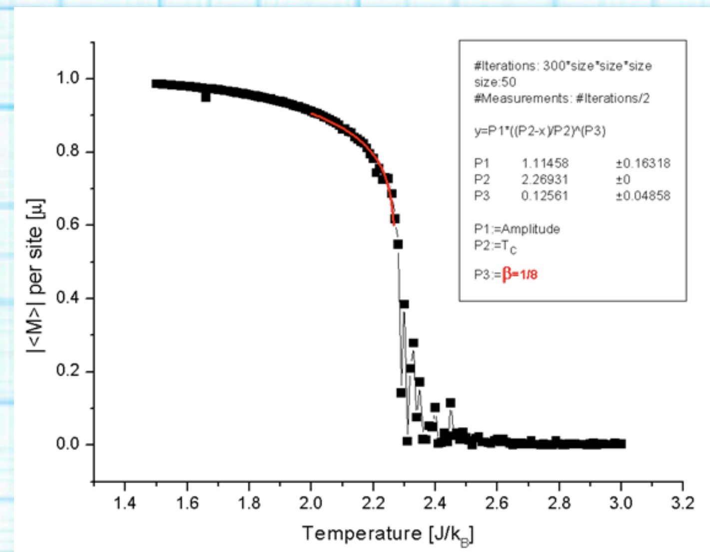
Snapshots of simulations of an 2d Ising model

What is so special about scale invariance ?

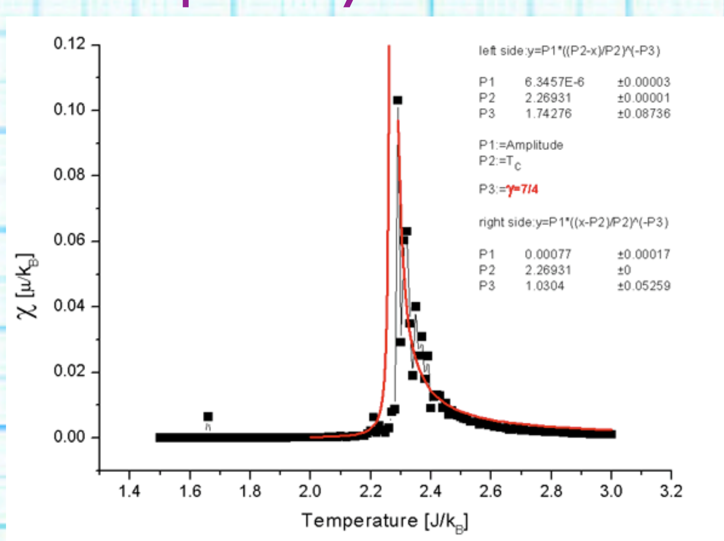
Equilibrium critical phenomena
The 2 dim Ising model



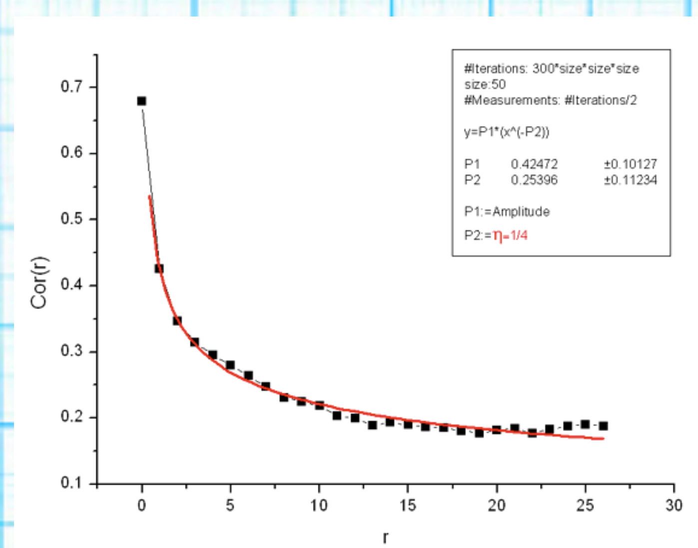
Order parameter



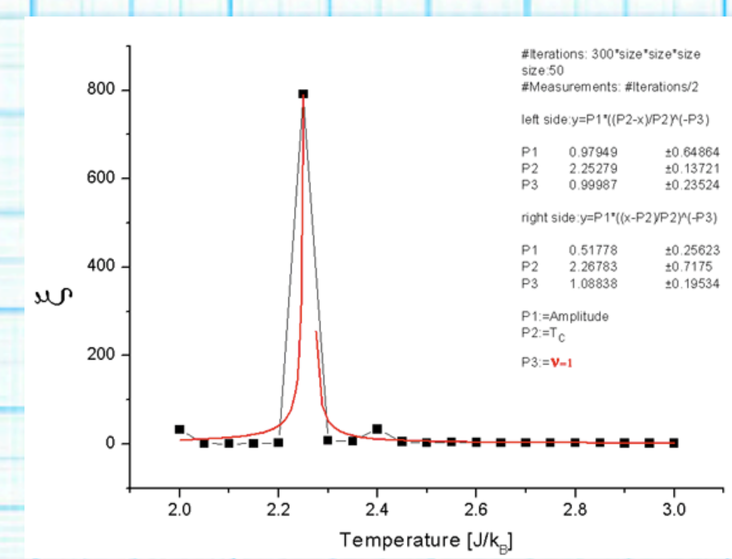
Susceptibility



Correlation function



Correlation length



Figures from: ***The Phase Transition of the 2D-Ising Model*** **by Lilian Witthauer and Manuel Dieterle**
http://quantumtheory.physik.unibas.ch/bruder/Semesterprojekte2007/p1/index.html#nameddest=x1-110002.1.6

Criticality - self-similarity

- No characteristic scales in space nor in time
- Interdependence throughout the system
- Ideally studied in terms of correlations functions

(or other information theoretic measures: mutual information, transfer entropy, etc.)

What to probe?

- Focus on correlation functions

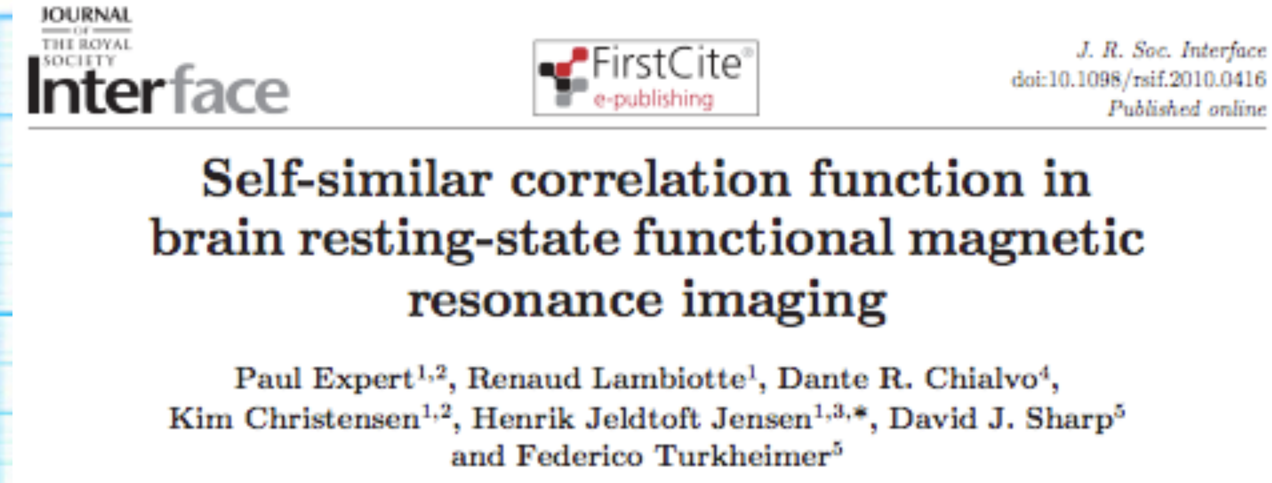
$$C(\mathbf{r},t) = \langle A(\mathbf{r}_0,t_0)A(\mathbf{r}_0+\mathbf{r},t_0+t)\rangle_{\mathbf{r_0},t_0} - \langle A(\mathbf{r}_0,t_0)\rangle^2$$

Event analysis

- Identify control parameter (humidity, background activity)

  Plot event sizes versus control parameter

Brain, Bold and Correlations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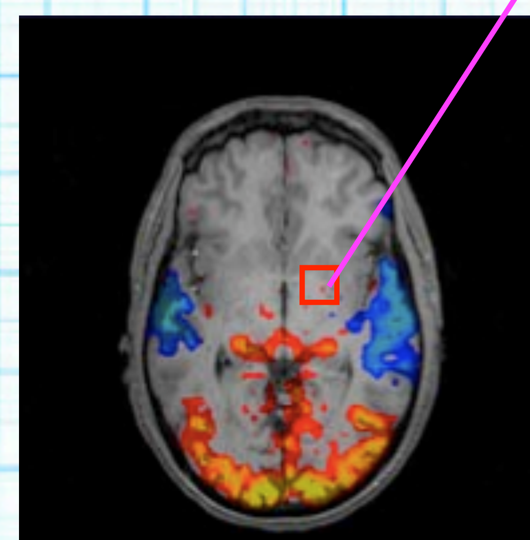
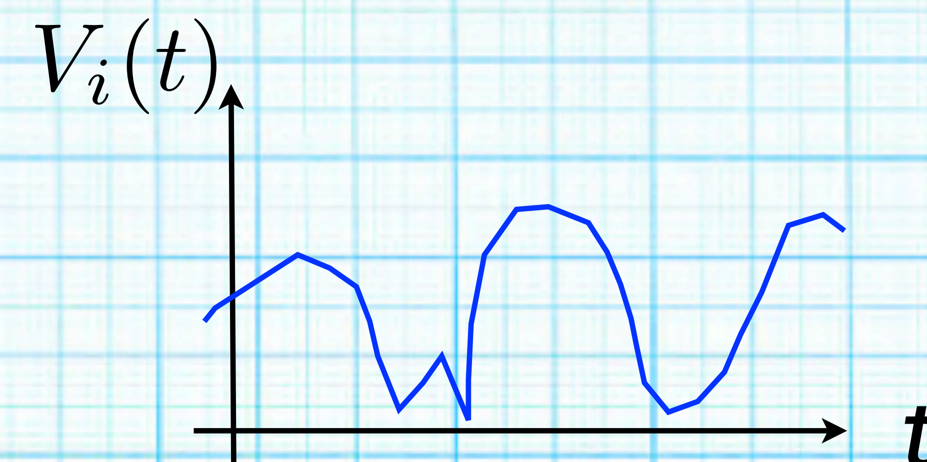
J. R. Soc. Interface
doi:10.1098/rsif.2010.0416
Published online

Self-similar correlation function in brain resting-state functional magnetic resonance imaging

Paul Expert[1,2], Renaud Lambiotte[1], Dante R. Chialvo[4], Kim Christensen[1,2], Henrik Jeldtoft Jensen[1,3,*], David J. Sharp[5] and Federico Turkheimer[5]

Rest state fMRI correlations





Correlation matrix

$$C(i,j) = C(|\mathbf{r}_i - \mathbf{r}_j|) = \frac{\langle V_i V_j \rangle - \langle V_i \rangle \langle V_j \rangle}{\sigma_i \sigma_j}$$

fMRI correlation functions

Analysis of scale invariance

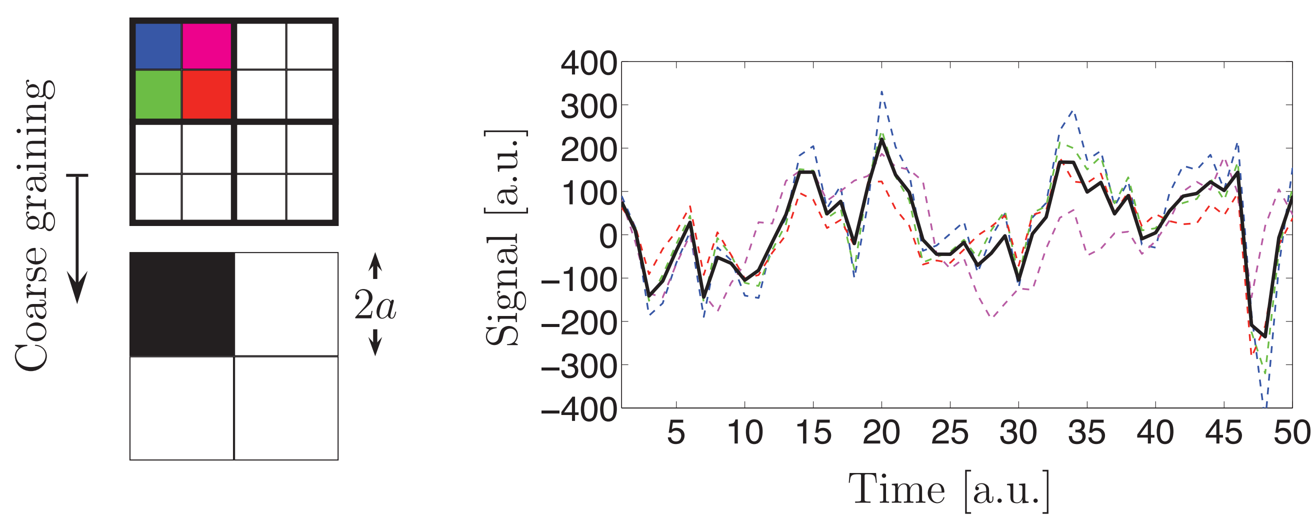


FIG. 6: (Left) Example of coarse graining in 2 dimensions where there are 4 boxes $\mathcal{B}$ within a block-box $\mathcal{B}'$. (Right) The four dashed-colored signals from the four original boxes $\mathcal{B}$ are averaged to produce the solid-black coarse-grained signal of $\mathcal{B}'$.

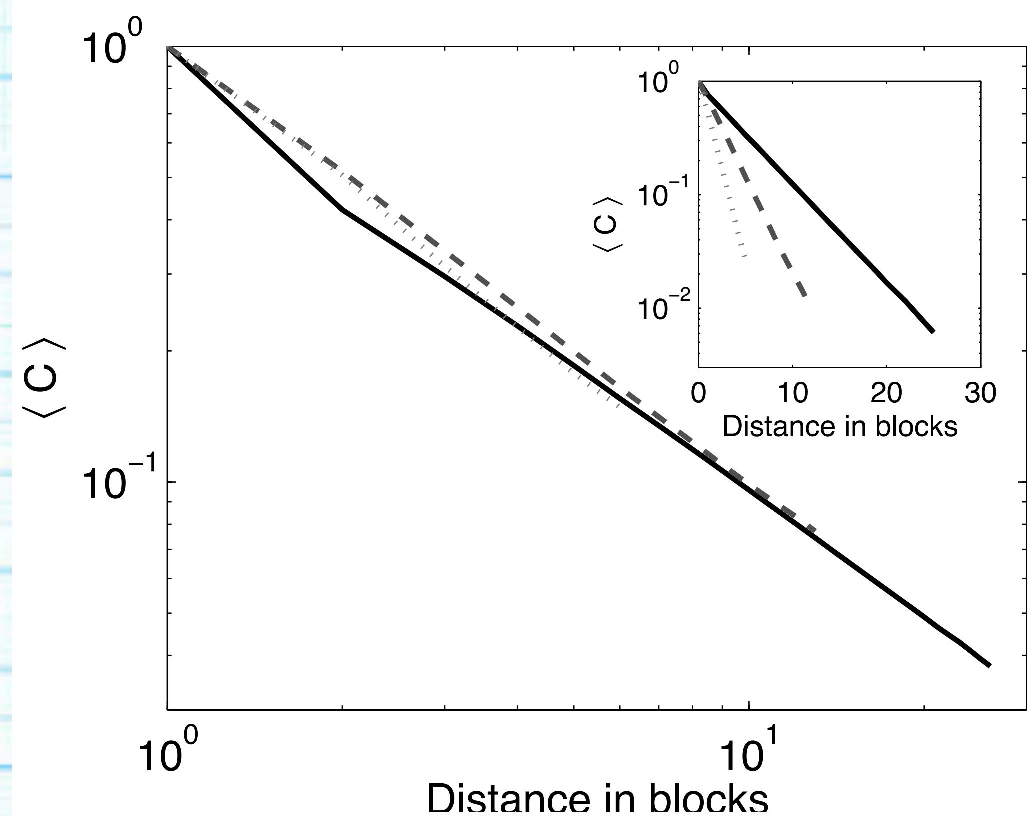


FIG. 7: Main: The average correlation function $\langle C \rangle$ vs the block distance for three coarse-graining levels of an hypothetical object with a power-law form of the correlation function. The graphs collapse in voxel distance. Inset: Results from an hypothetical object having an exponential form of the correlation function where the curves do not collapse in block distance.

fMRI correlation functions

All $V_i(t)$

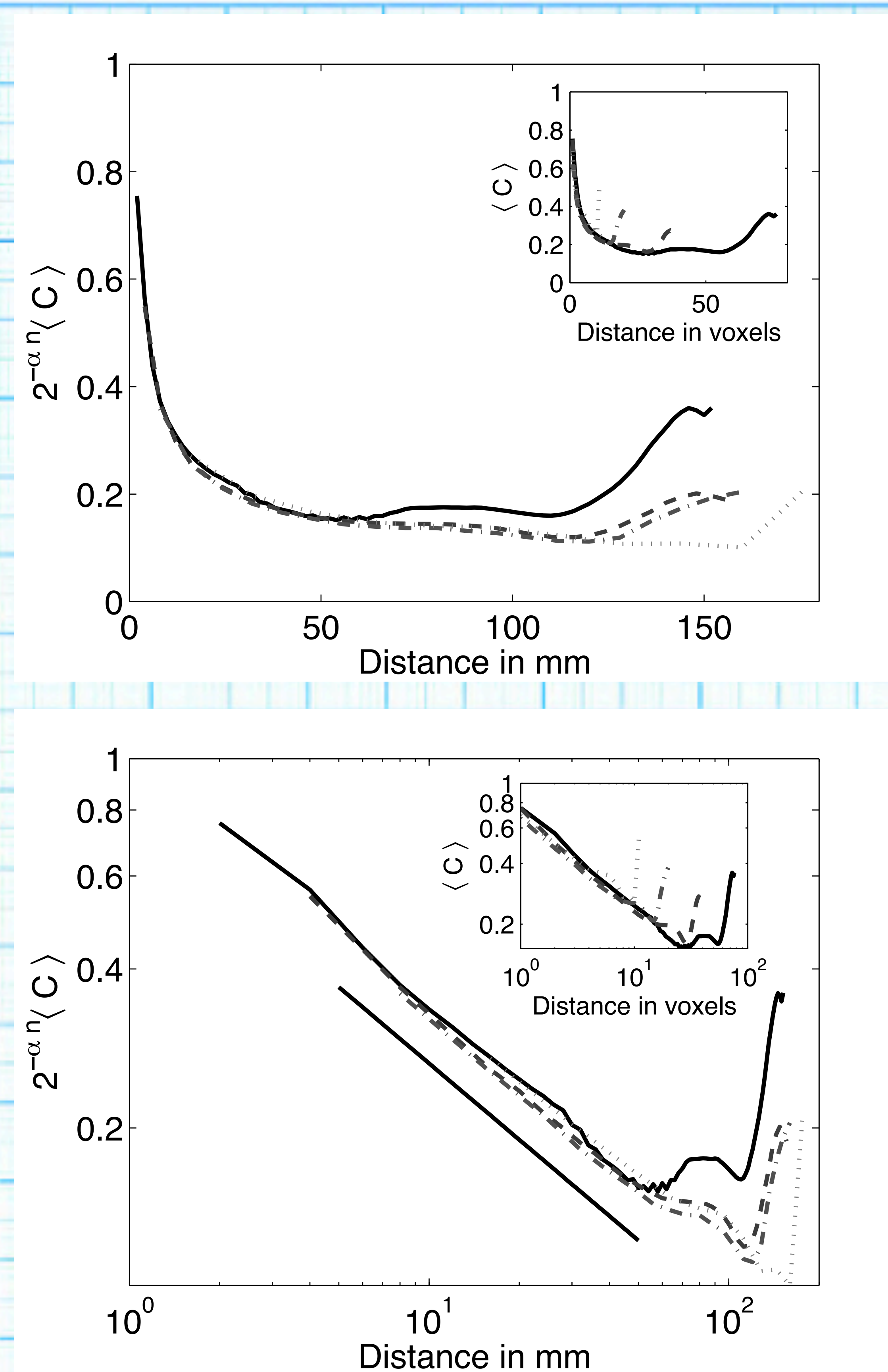


FIG. 1: The renormalized average correlation function $2^{-\alpha n}\langle C\rangle$ vs real distance (inset: $\langle C\rangle$ vs voxel distance) for the four levels of description for one subject. Full line: $128 \times 128 \times 31$ $(n = 0)$, dashed line: $64 \times 64 \times 16$ $(n = 1)$, dashed-dotted line: $32 \times 32 \times 8$ $(n = 2)$ and dotted line: $16 \times 16 \times 4$ $(n = 3)$. Top panel linear-linear and bottom panel log-log axis, respectively. Full straight line is a guide to the eye for a power law with exponent $\beta = 0.47 \pm 0.2$. Parameter $\alpha = 0.45$.

Paul Expert & Renaud Lambiotte

fMRI correlation functions

Temporal aspects

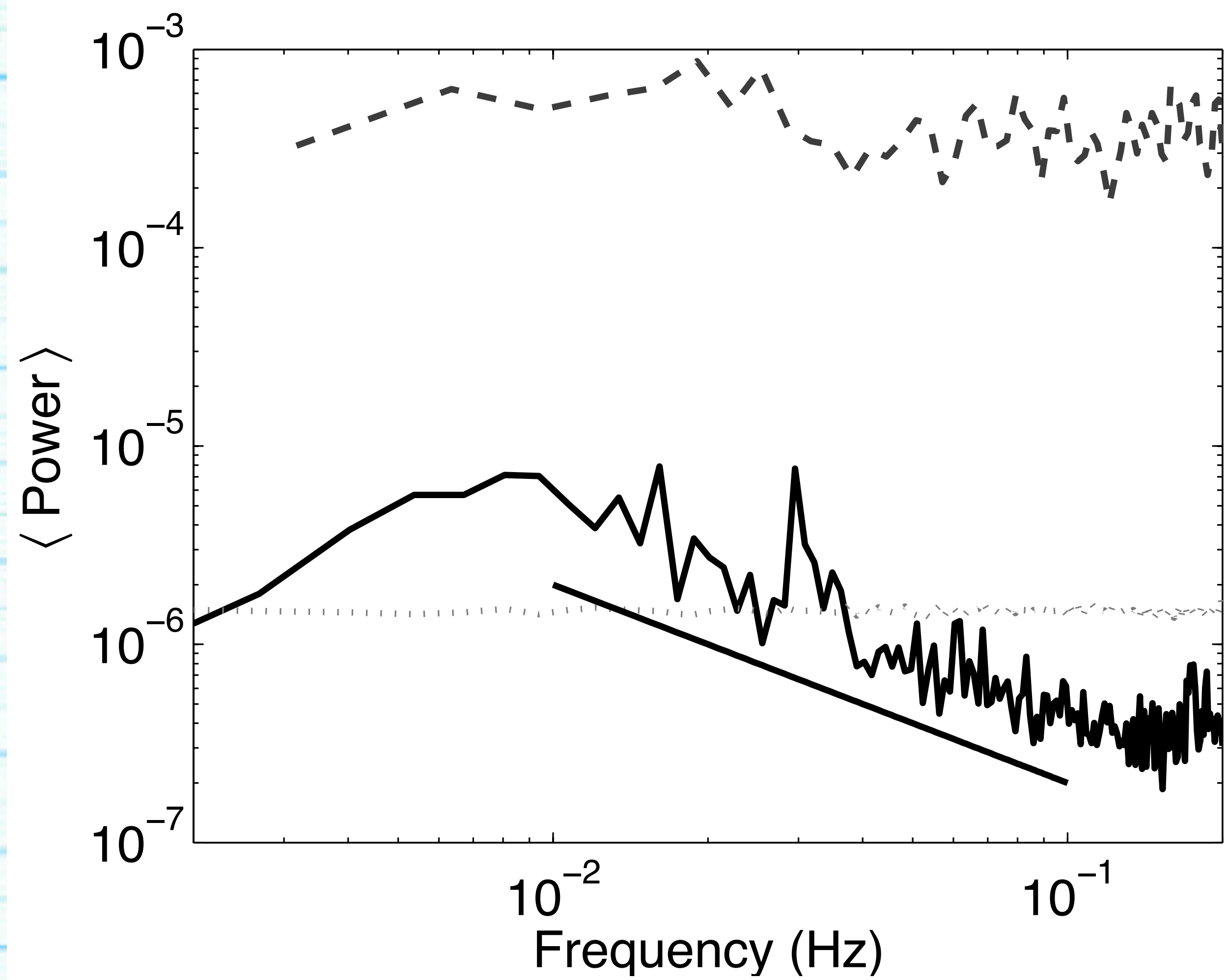


FIG. 5: Average power spectrum for the fMRI time series (solid line), the phantom time series (dashed line) and the randomized time series (dotted line). Full straight line is a guide to the eye for a $1/f$ spectrum.

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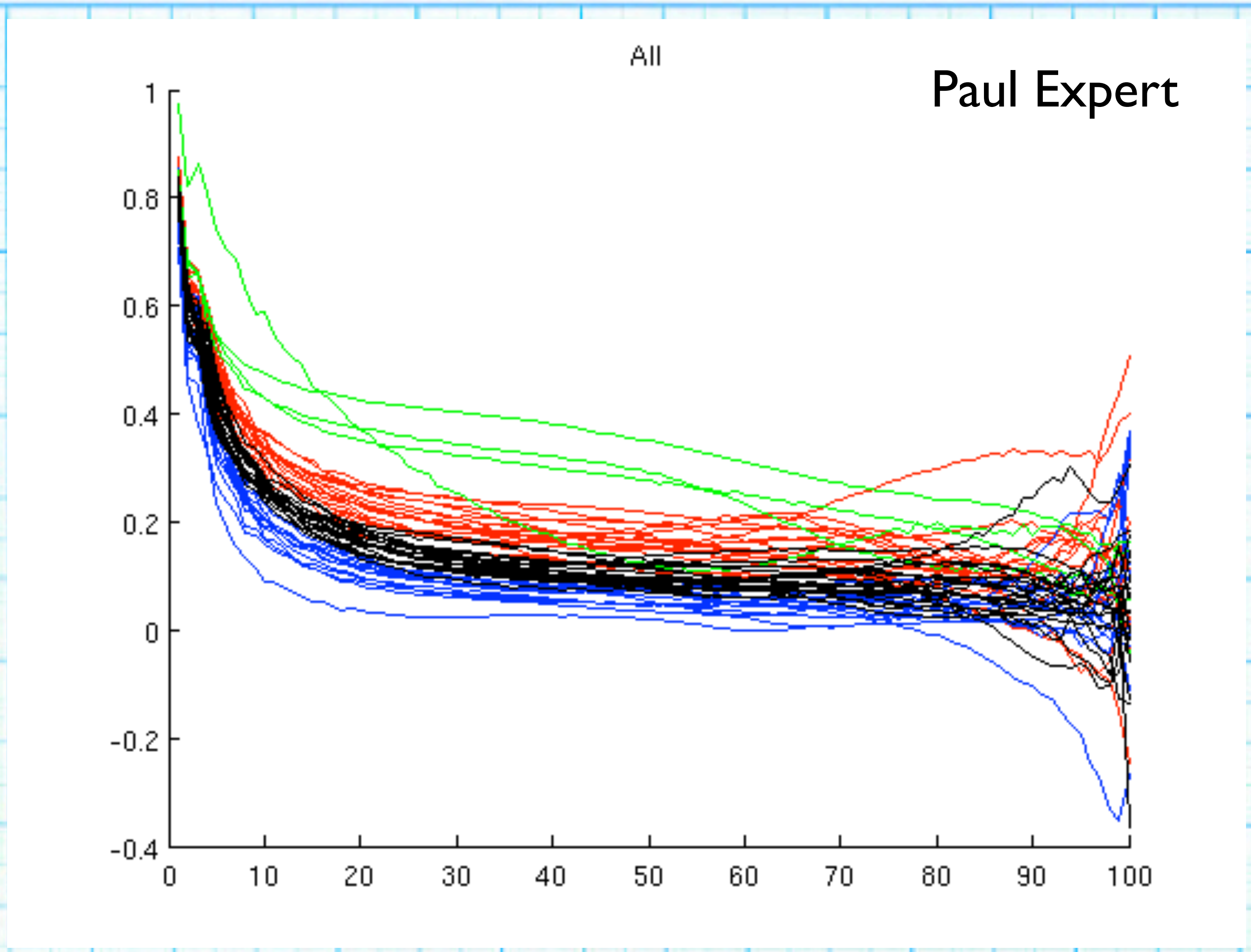
Generality



Correlation functions of 53 prematurely born babies hierarchically grouped together to form clusters.

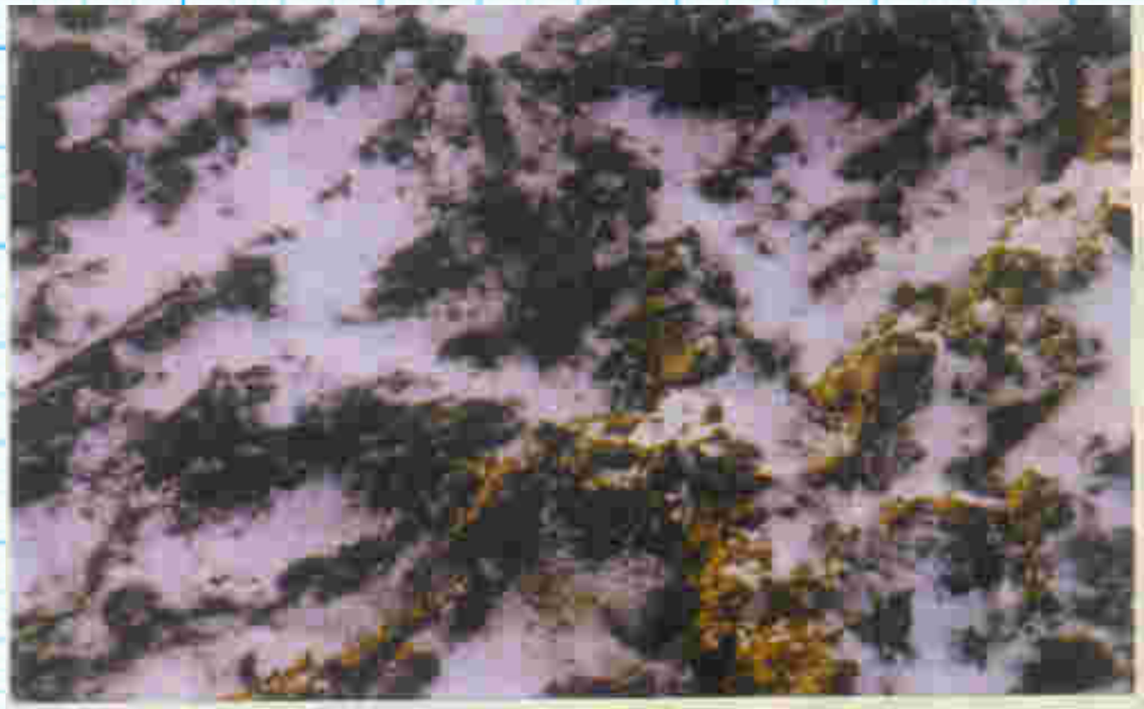
Data from Valentina Doria and Professor David Edwards
Neonatal Medicine, MRC Clinical Sciences Centre
Hammersmith Hospital, Imperial College London

SOC or avalancheology

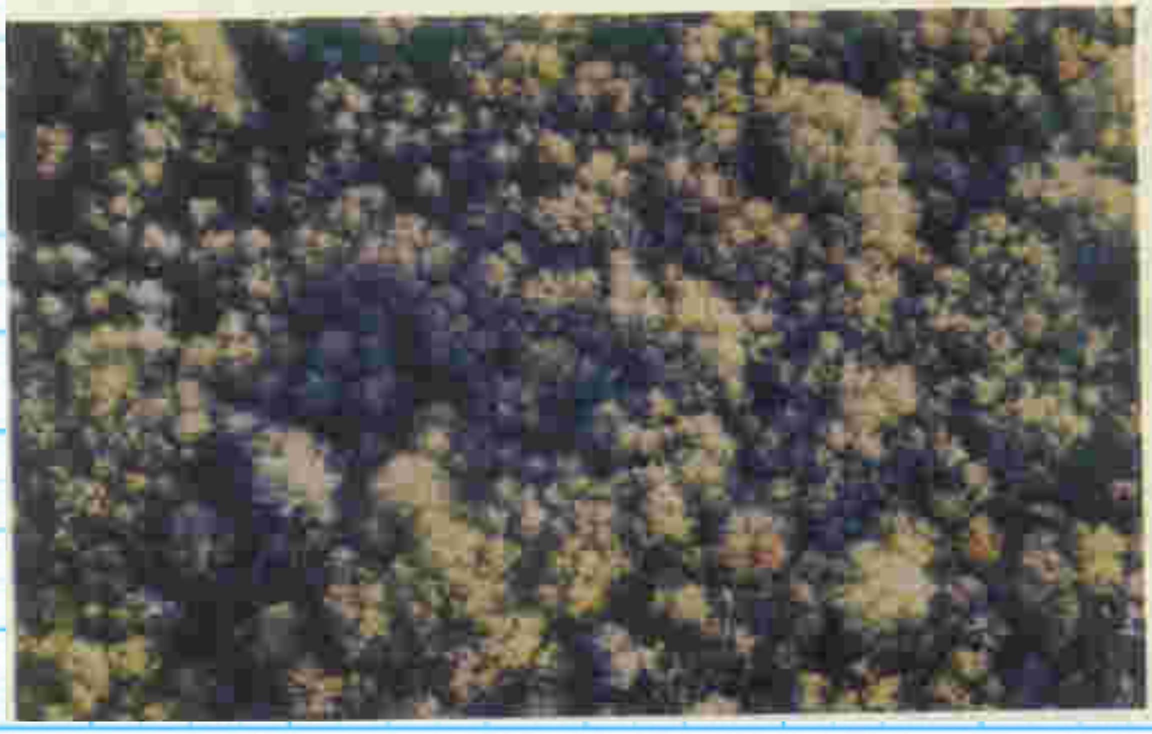
Scale free behaviour out of equilibrium

Spatial fractals

- Clouds
- Mountains
- Cauliflower
-



Snow on ground



Canopy

An explanation needed!

If critical behaviour is so common there must surely be one universal mechanism behind (? ?)

PHYSICAL REVIEW LETTERS

VOLUME 59 | 27 JULY 1987 | NUMBER 4

Self-Organized Criticality: An Explanation of $1/f$ Noise

Per Bak, Chao Tang, and Kurt Wiesenfeld
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or $1/f$ noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

Many more models:

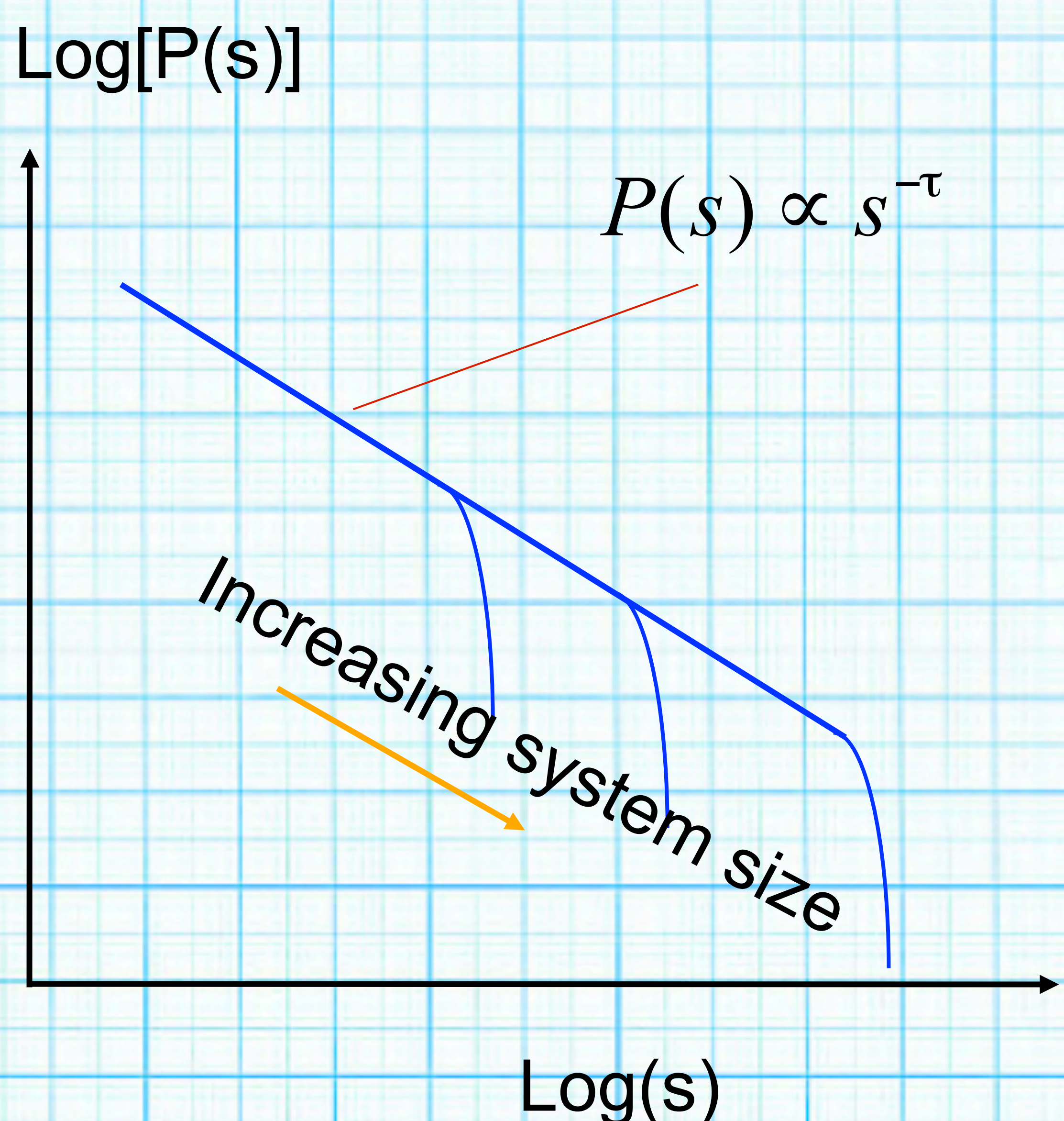
- Earth quake model (Olami-Feder-Christensen)
- Forest fires or epidemics (Drossel-Schwabl)
- Deterministic lattice gas (Jensen)

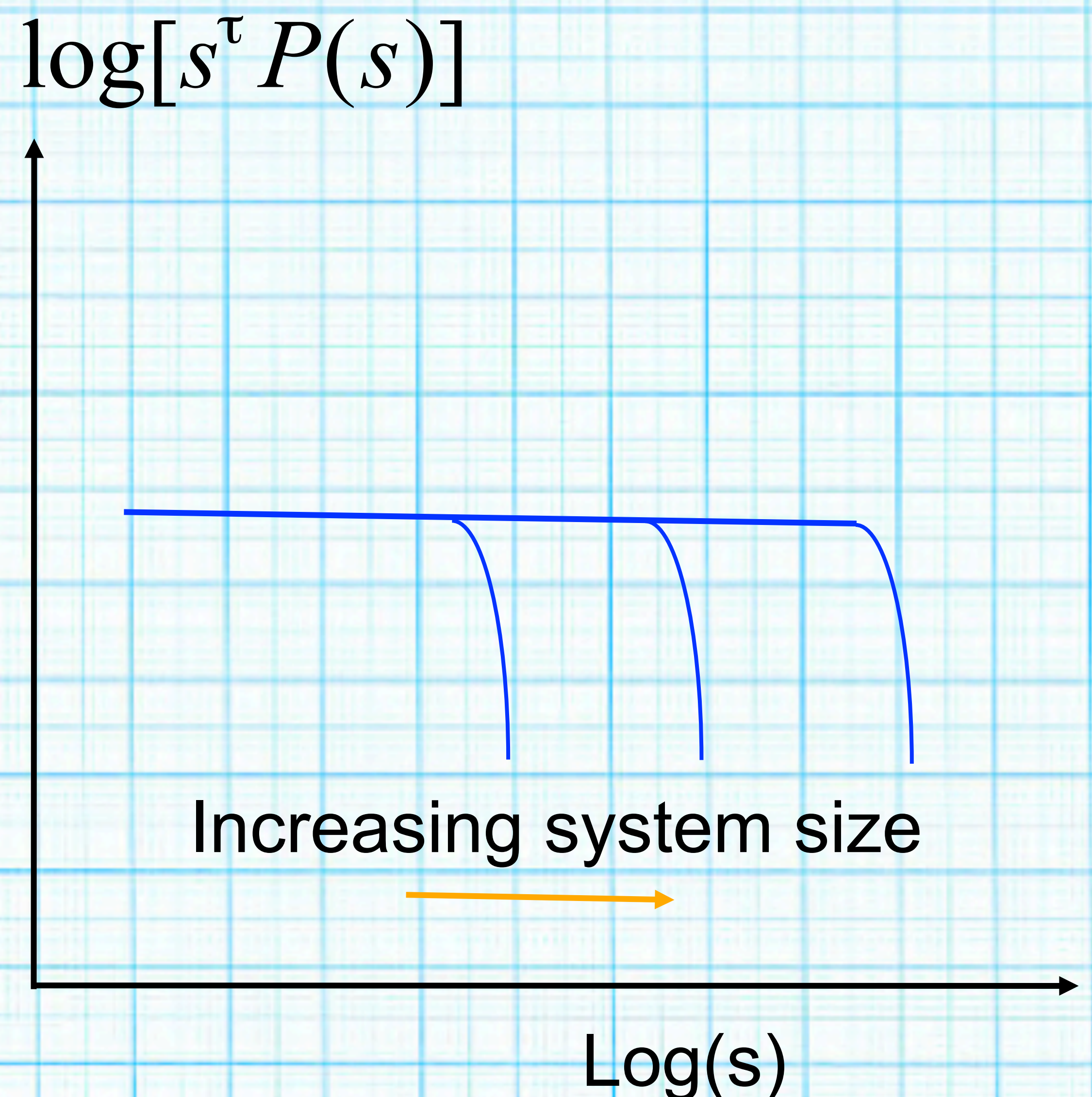
All exhibit scale invariance in the form of power laws for the distributions of events or avalanches and the DLG has 1/f.

Well, at least to some degree

Broken scaling

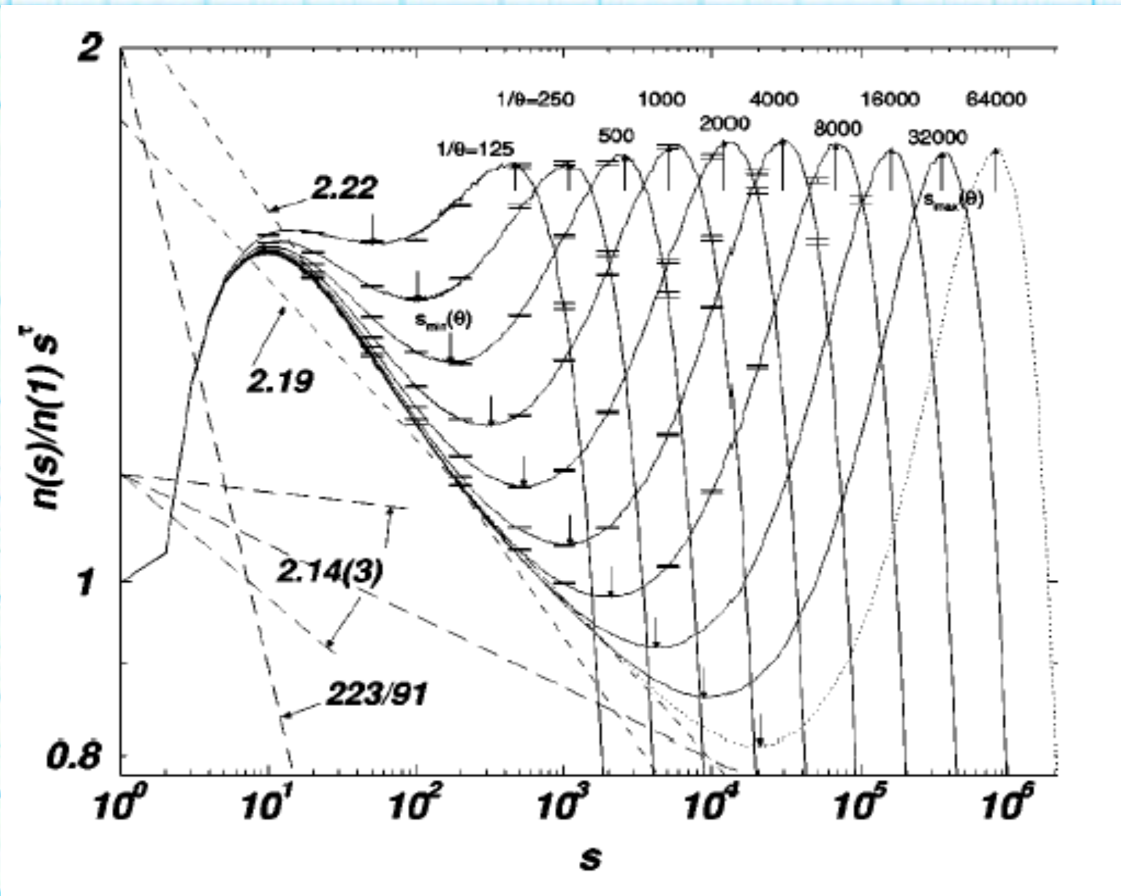
The ideal situation





The real situation

The Drossel-Schwabl forest fire model



From G. Prussner & HJJ,

Phys. Rev. E. **65**, 056707 (2002).

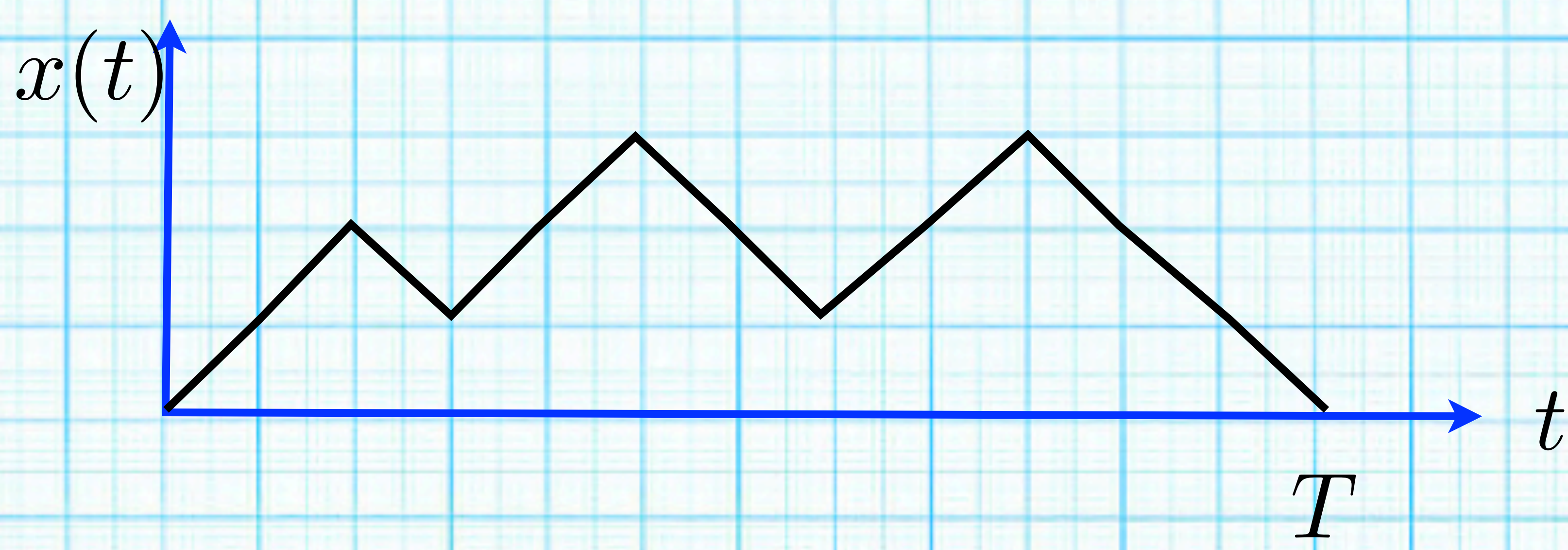
See also Grassberger.

Why: the dynamics keeps moving the system away from criticality

3/2

The random walker as a non-critical power law generator

$$x(t+1) = x(t) + \chi \qquad \text{with} \qquad \chi \in \{-1, 1\}$$



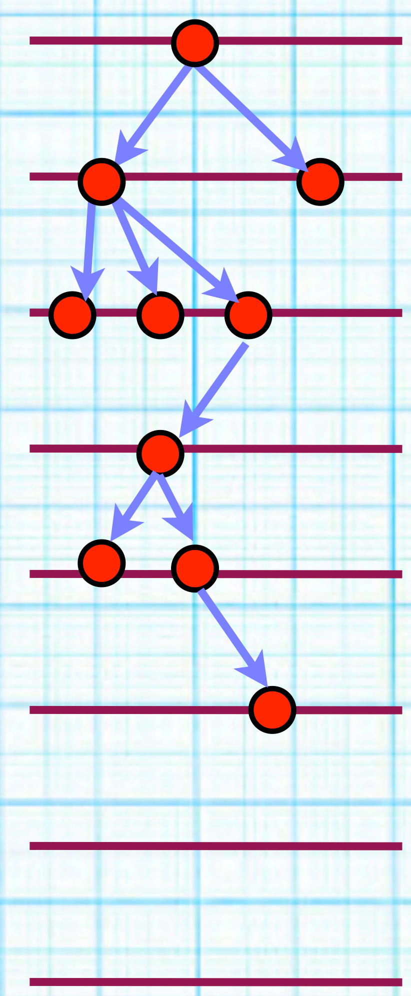
$$P(T) \sim T^{-3/2}$$

$$\langle (x(t+T) - x(t))^2 \rangle \sim T$$

Consider the displacement at time t $\quad \Delta x(t) = x(t+1) - x(t)$

$$\langle \Delta x(t) \Delta x(t+T) \rangle = \delta(T)$$

The branching process



Branching probability p_n

Average branching ratio $\sigma = \sum_n n p_n$

Tree size distribution

$$P(S) \sim S^{-3/2} \exp(-S/S_c)$$

where $S_c \to \infty$ for $\sigma \to 1$

Proper many body criticality only if space and time exhibit scale invariance / power laws

The rain and brain approach

Henrik Jeldtoft Jensen

Peters & Neelin, Nature Physics, 2006

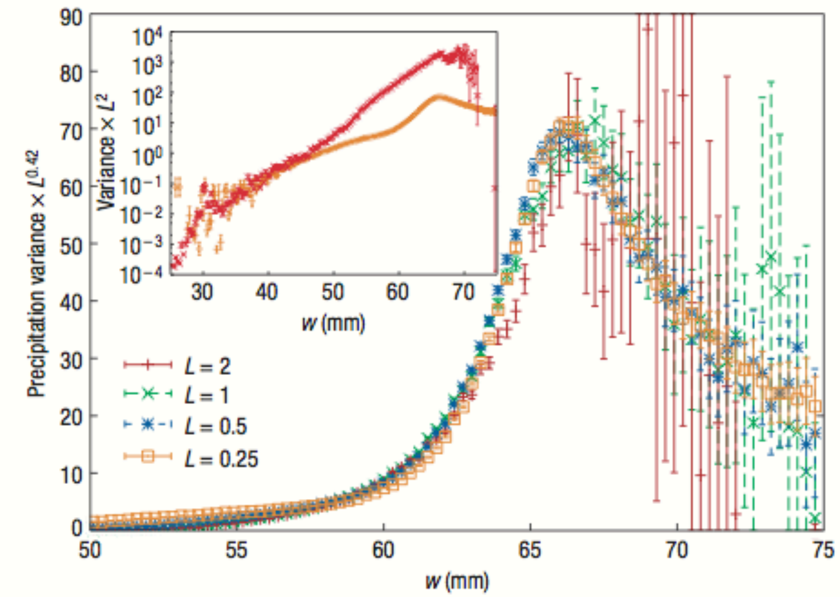


Figure 2 Finite-size scaling. The variance of the order parameter $\sigma_P^2(w)$ as a function of w, rescaled with $L^{0.42}$ for system sizes 0.25°, 0.5°, 1° and 2° in the western Pacific. From $w \approx 57$ mm, this produces a good collapse. The inset shows that away from the critical point, up to $w \approx 40$ mm a trivial rescaling with $L^{d=2}$ works adequately. This suggests that the non-trivial collapse is indeed a result of criticality. The error bars are standard errors, determined through the zeroth, second and fourth moments of the distribution of P at any given w. Measurements of $P(w)$ are considered independent, which holds well between satellite overpasses, although not within individual tracks.

Henrik Jeldtoft Jensen

Peters & Neelin, Nature Physics, 2006

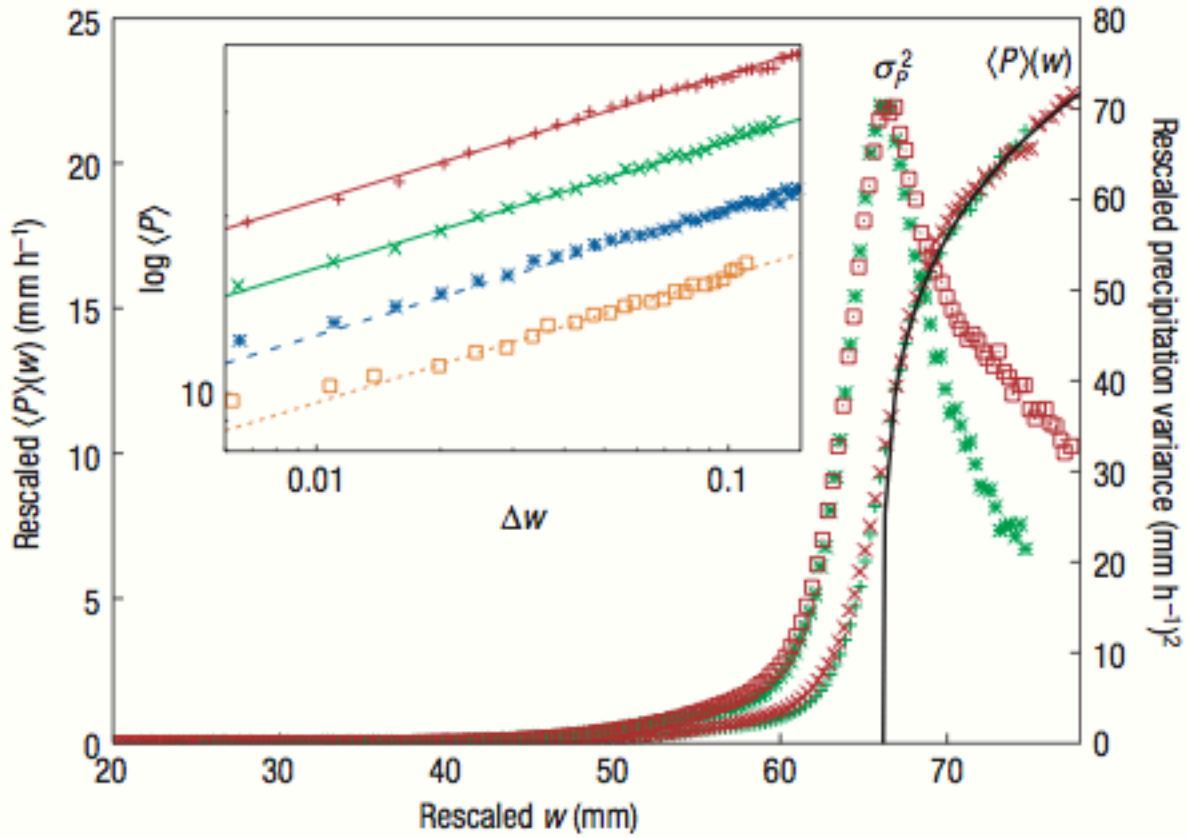
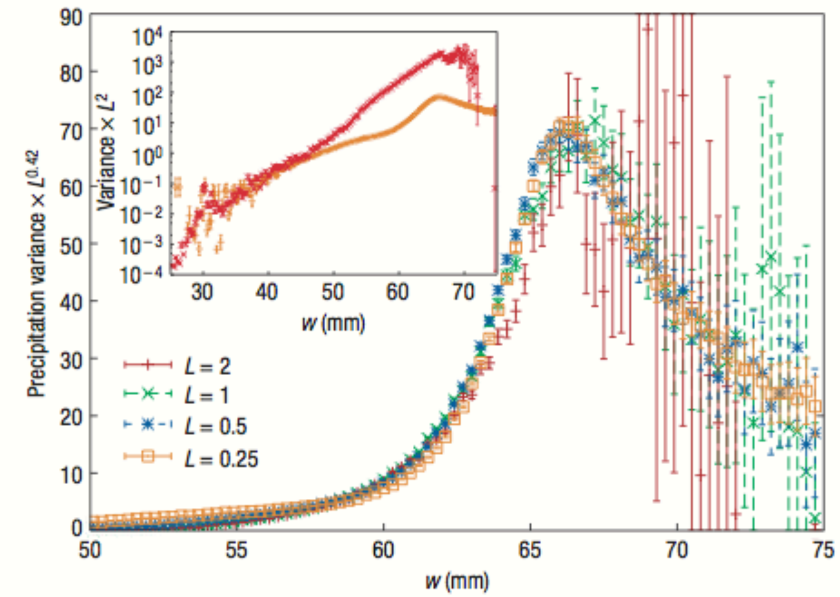


Figure 1 Order parameter and susceptibility. The collapsed (see text) precipitation rates $\langle P \rangle(w)$ and their variances $\sigma_P^2(w)$ for the tropical eastern (red) and western (green) Pacific as well as a power-law fit above the critical point (solid line). The inset shows on double-logarithmic scales the precipitation rate as a function of reduced water vapour (see text) for western Pacific (green, 120E to 170W), eastern Pacific (red, 170W to 70W), Atlantic (blue, 70W to 20E), and Indian Ocean (pink, 30E to 120E). The data are shifted by a small arbitrary factor for visual ease. The straight lines are to guide the eye. They all have a slope of 0.215, fitting the data from all regions well.



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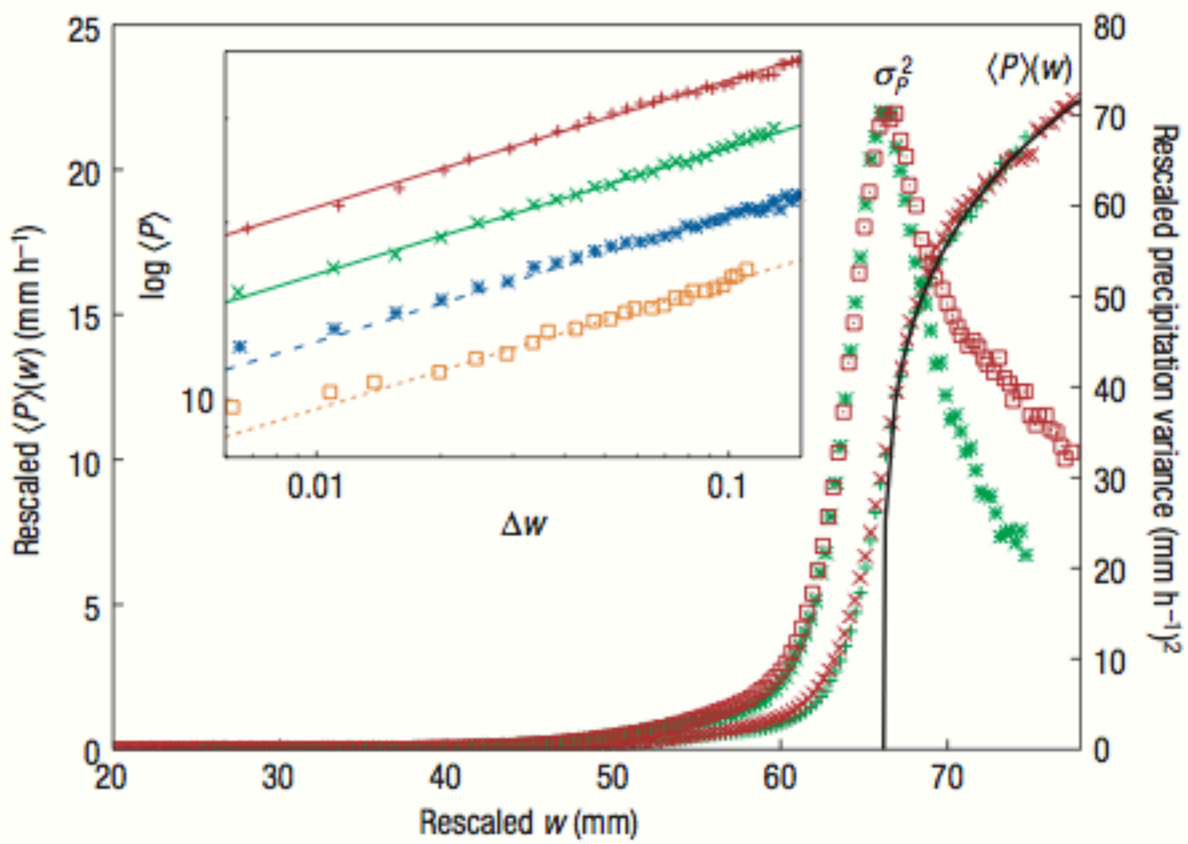

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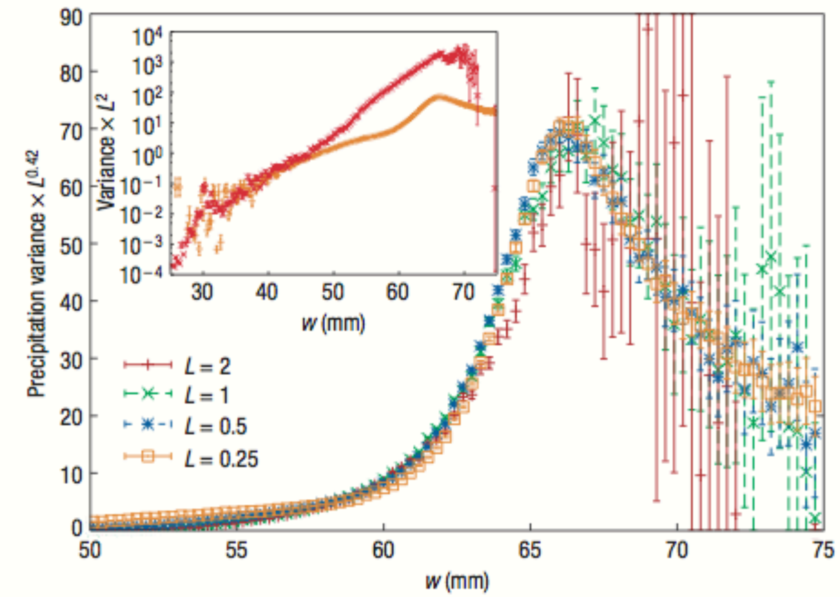

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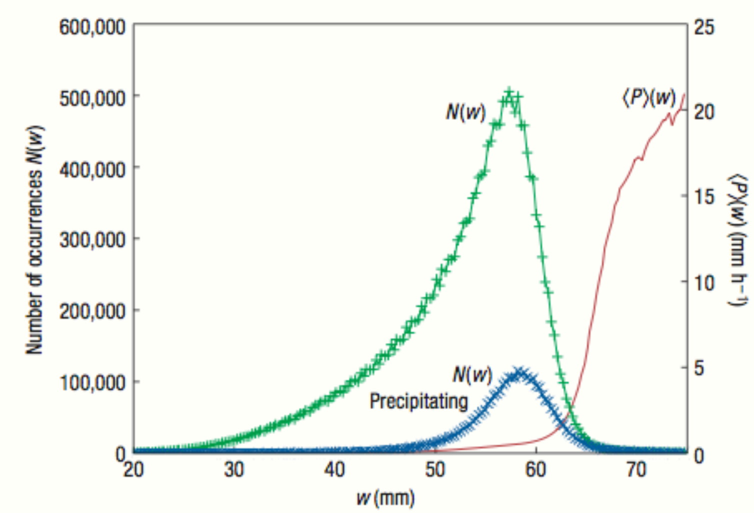
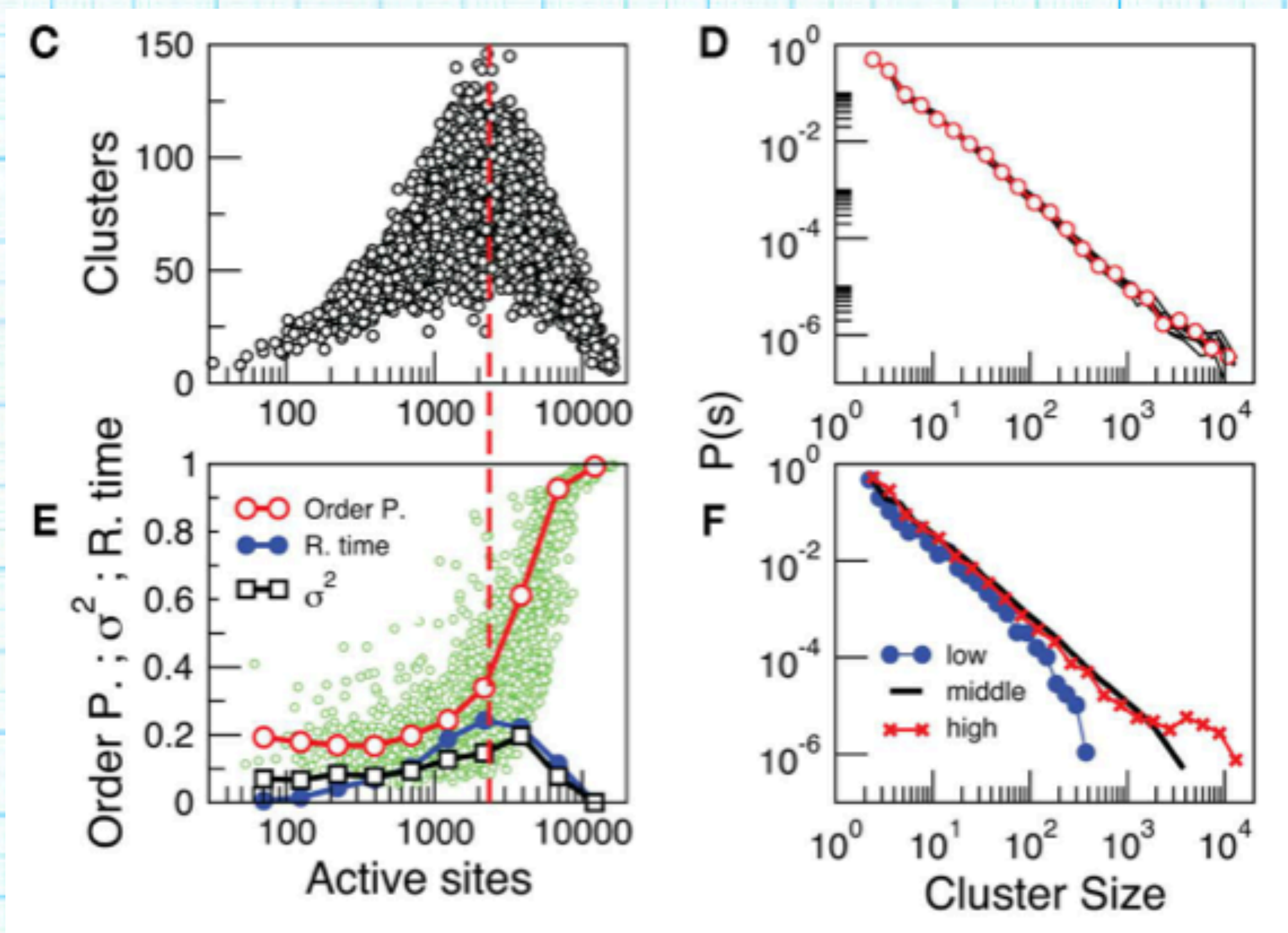


Figure 3 Residence times. The number of times $N(w)$ an atmospheric pixel of 0.25° × 0.25° was observed at water vapour w in the western Pacific, given an SST within a 1 °C bin at 30 °C. The green and blue lines show residence time for all points and precipitating points, respectively. The red line shows the order-parameter pick-up $\langle P\rangle(w)$ for orientation (precipitation scale on the right).

Brain



Dante Chialvo

of voxels) in one individual. **(C)** Instantaneous relation between the number of clusters vs. the number of active sites (i.e., voxels above the threshold) showing a positive/negative correlation depending whether activity is below/above a critical value [~2500 voxels, indicated by the dashed line here and in **(B)**]. **(D)** The cluster size distribution follows a power law spanning four orders of magnitude. Individual statistics for each of the ten subjects are plotted with lines and the average with symbols. **(E)** The order parameter, defined here as the (normalized) size of the largest cluster is plotted as a function of the number of active sites (isolated data points denoted by dots, averages plotted with circles joined by lines). The calculation of the residence time density distribution (R. time, filled circles) indicates that the brain spends relatively more time near the transition point (which corresponds to about 0.4 of the largest giant cluster observed). Notice that the peak of the R. Time in this panel coincides with the peak of the number of clusters in **(C)**. Note also that the variance of the order parameter (squares) increases as expected for a phase transition. **(F)** The computation of the cluster size distribution calculated for three ranges of activity (low: 0–800; middle: 800–5000; and high >5000) reveals the same scale invariance plotted in **(D)** for relatively small clusters, but shows changes in the cut-off for large clusters.

From: Tagliazucchi, et al.
frontiers in Physiology
Feb 2012

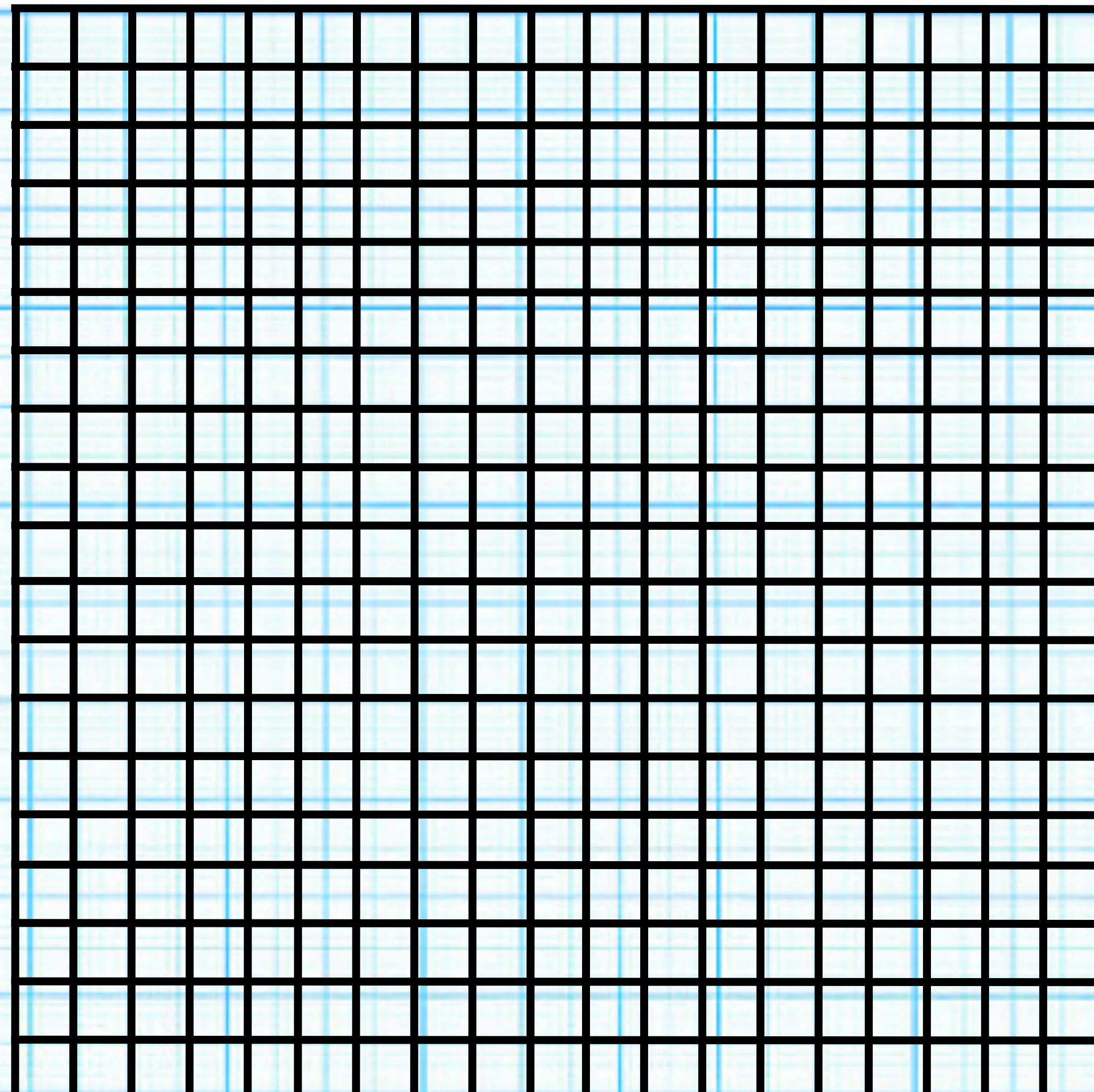
Question

What do we learn about the degree of criticality form this kind of analysis ?

Forest fire and percolation

Percolation

Percolation



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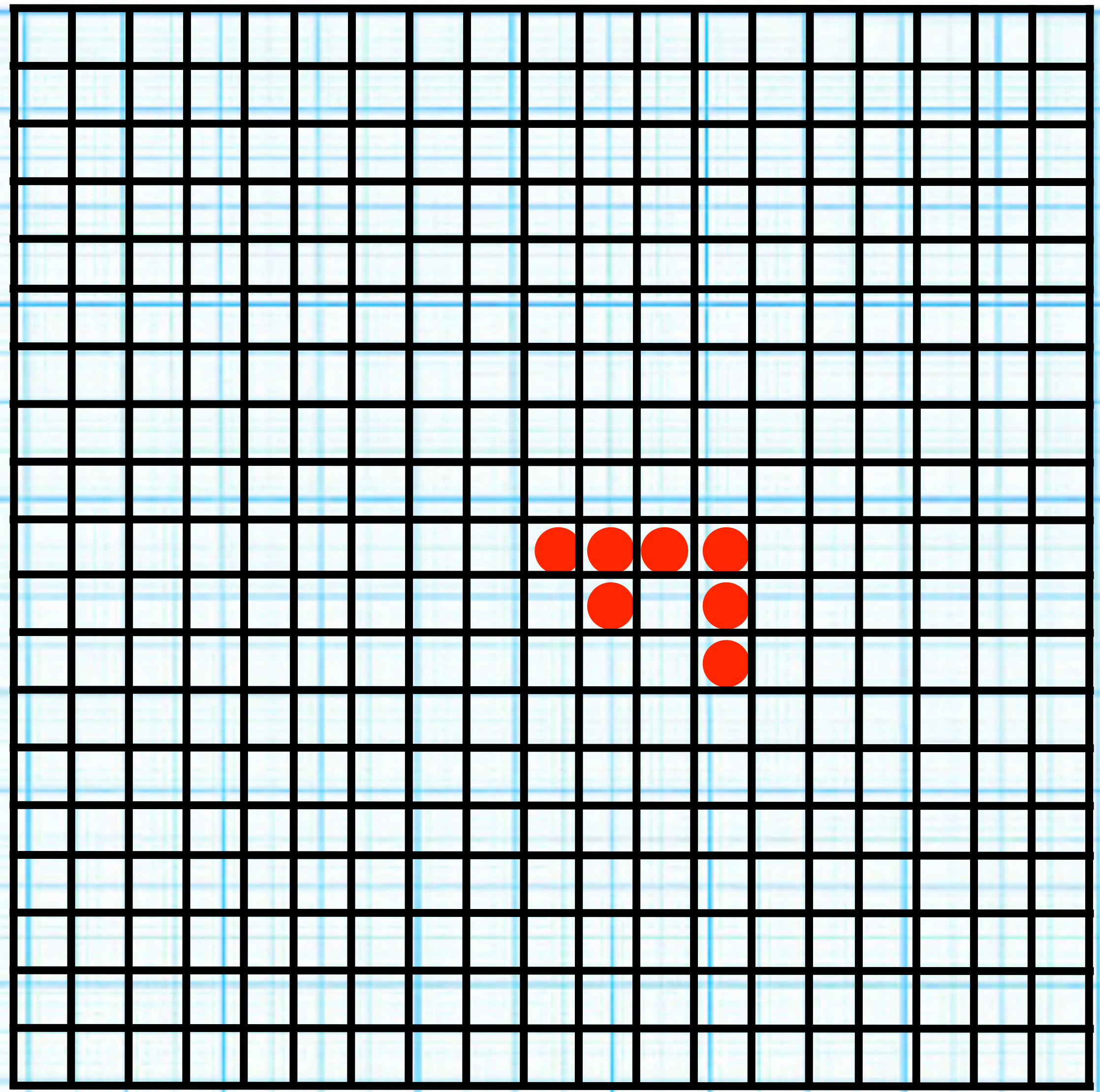

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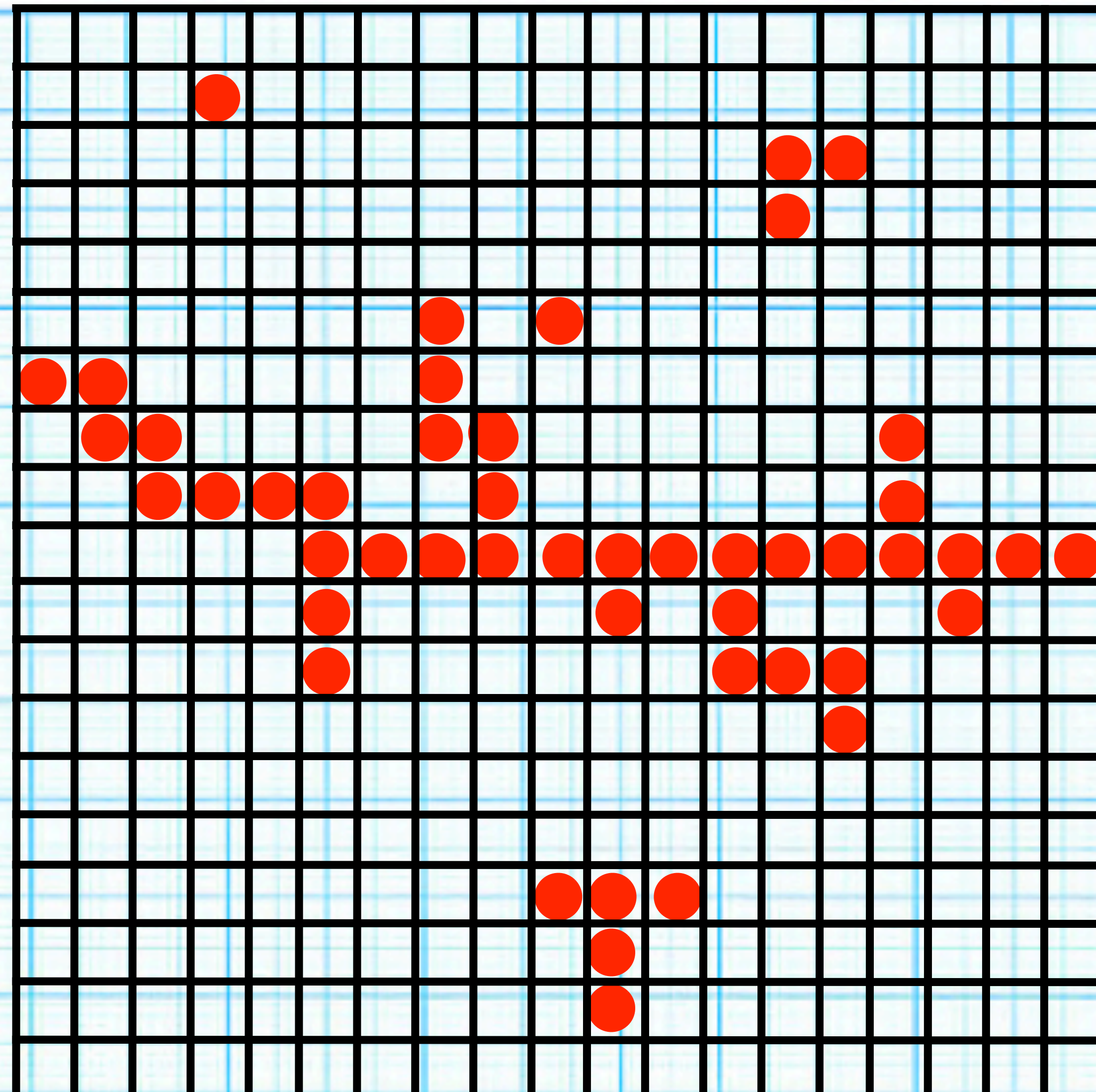

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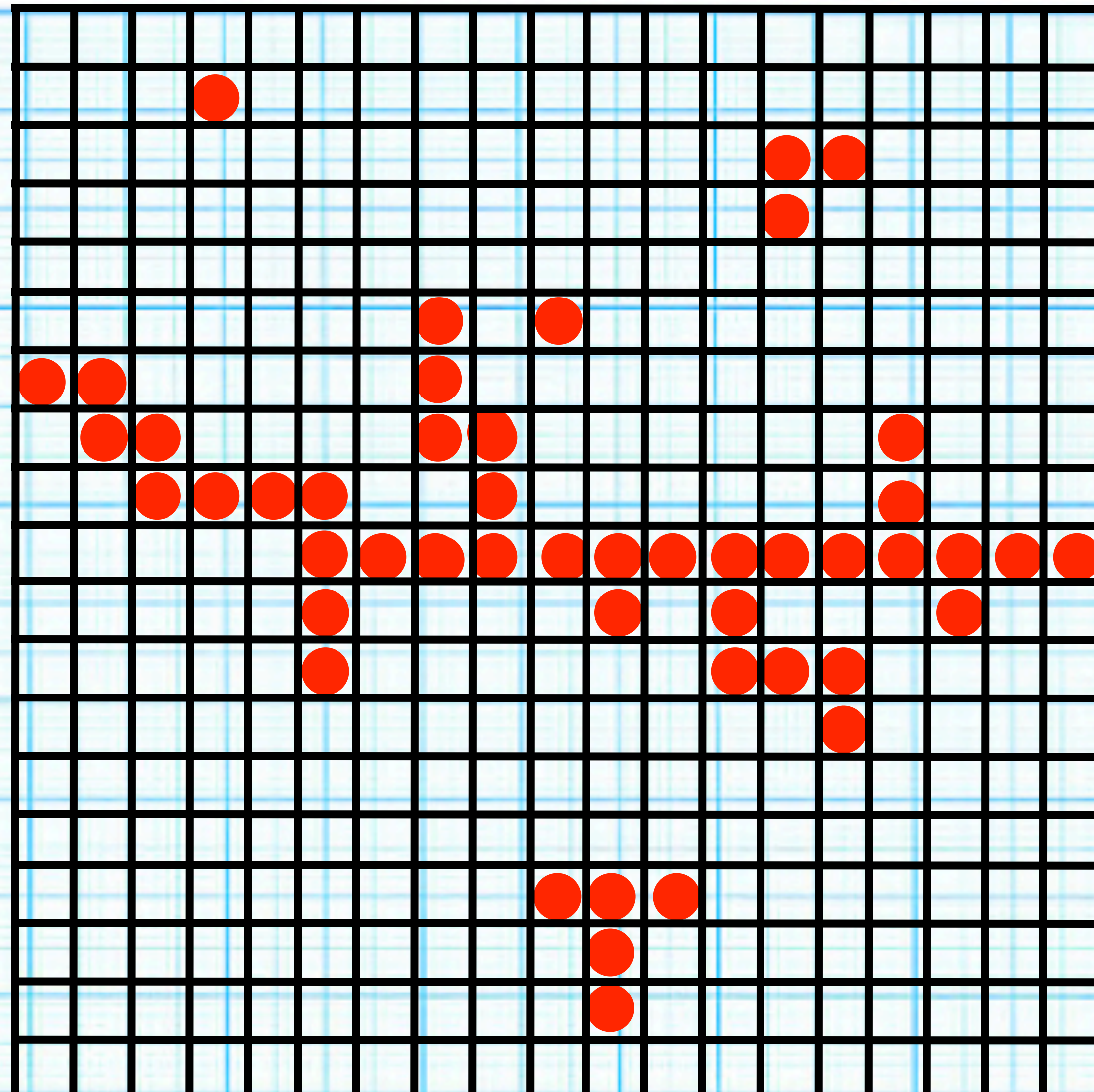

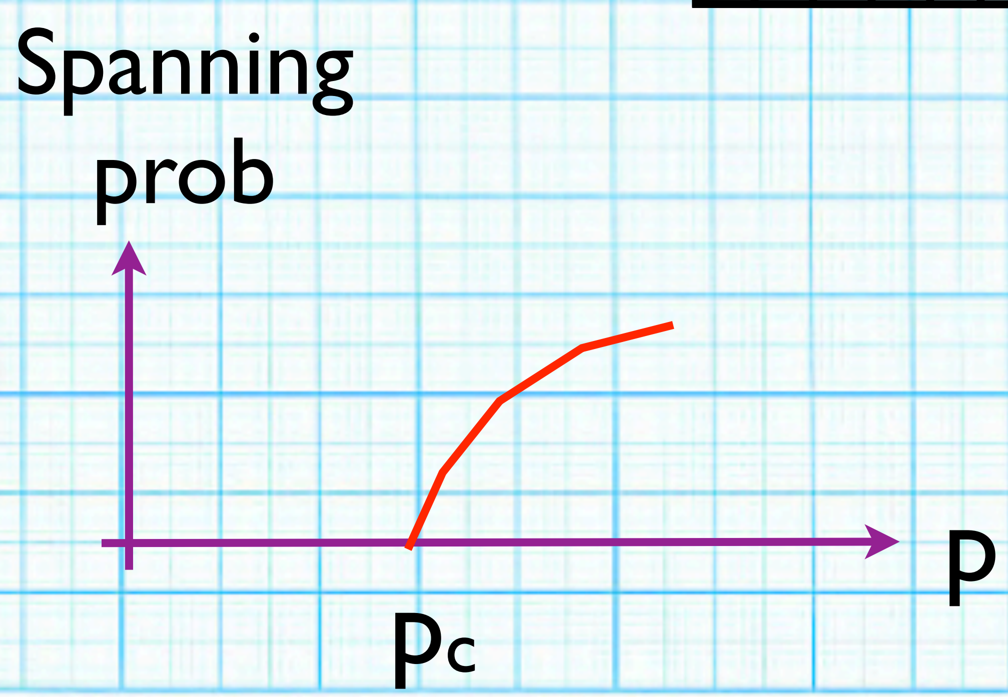

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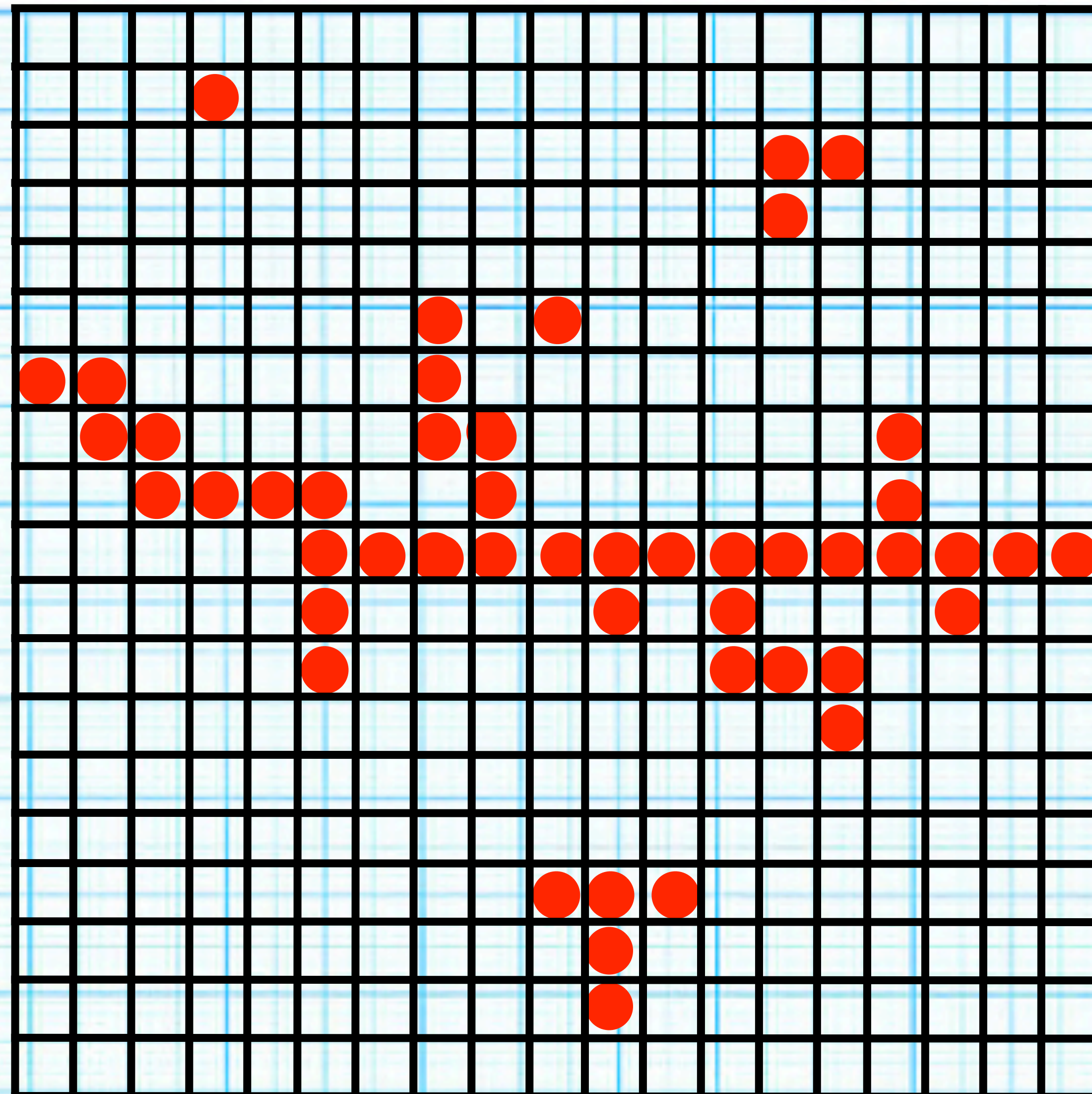
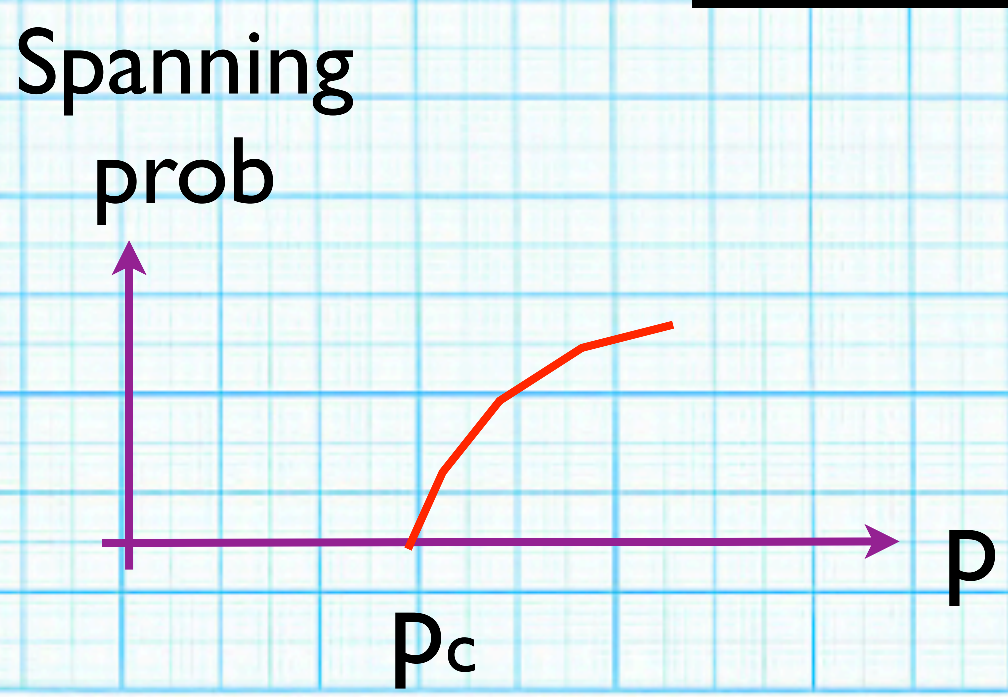
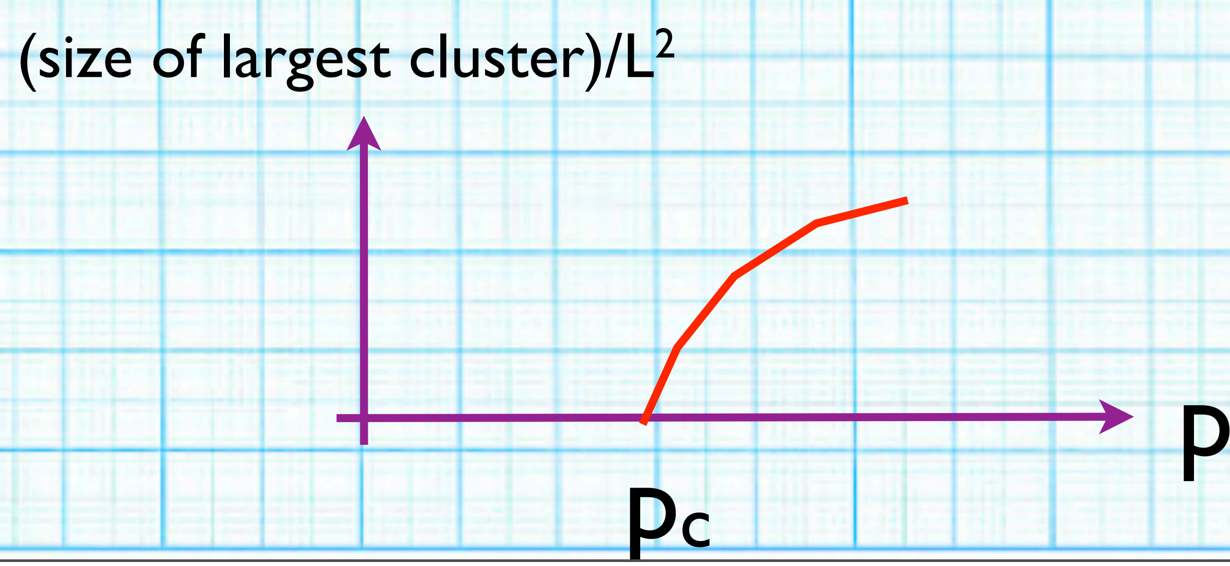
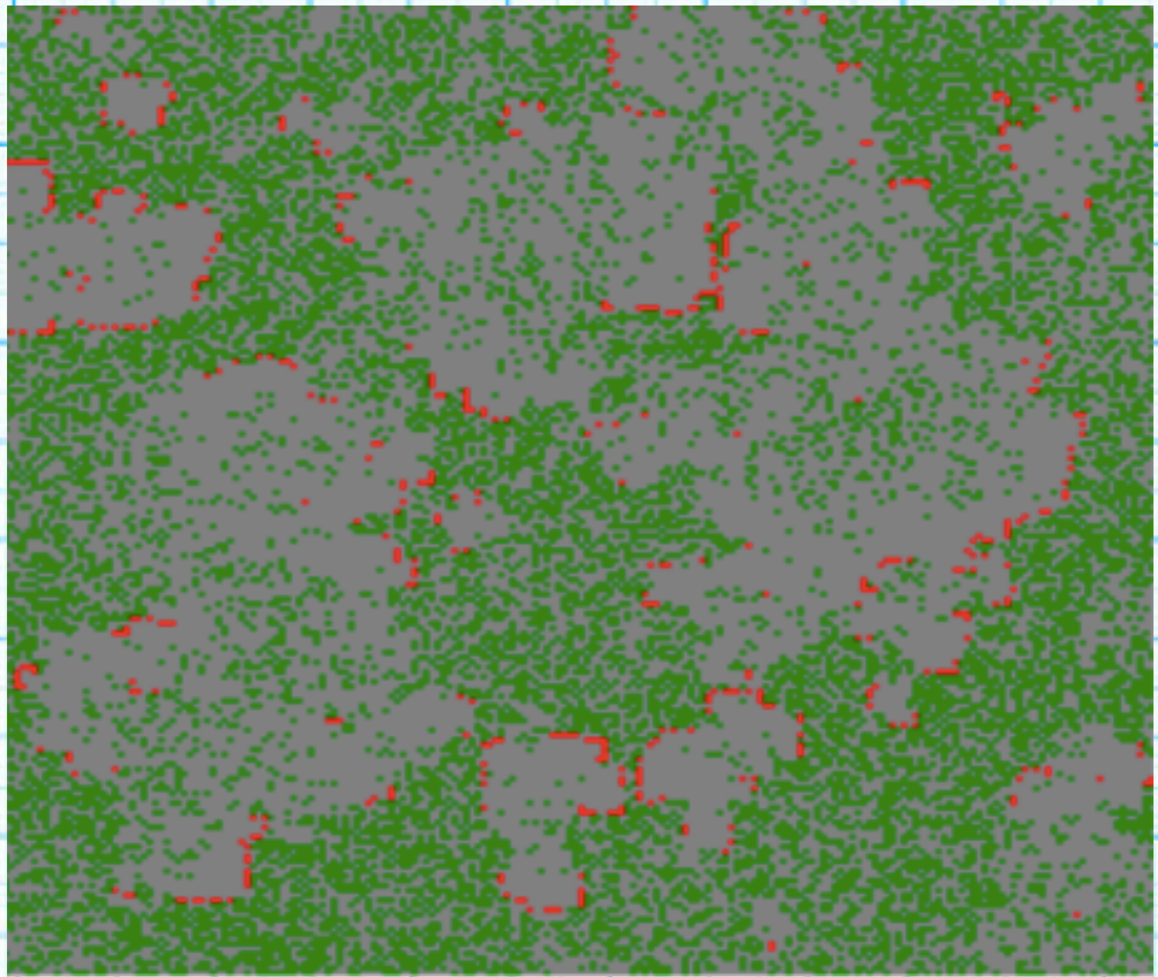






Forest fire model

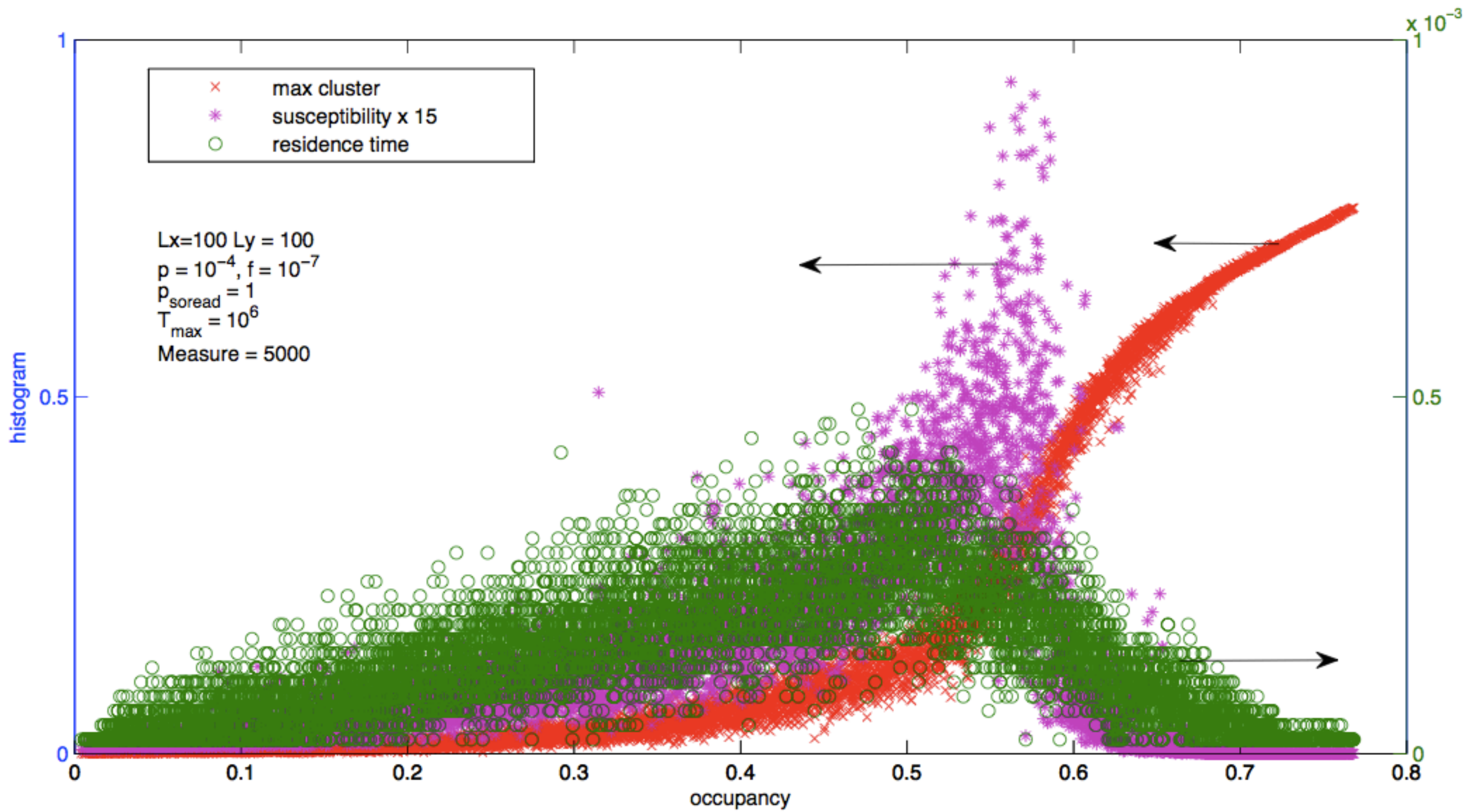
Drossel & Schwabl Phys Rev Lett 69, 1629 (1992)



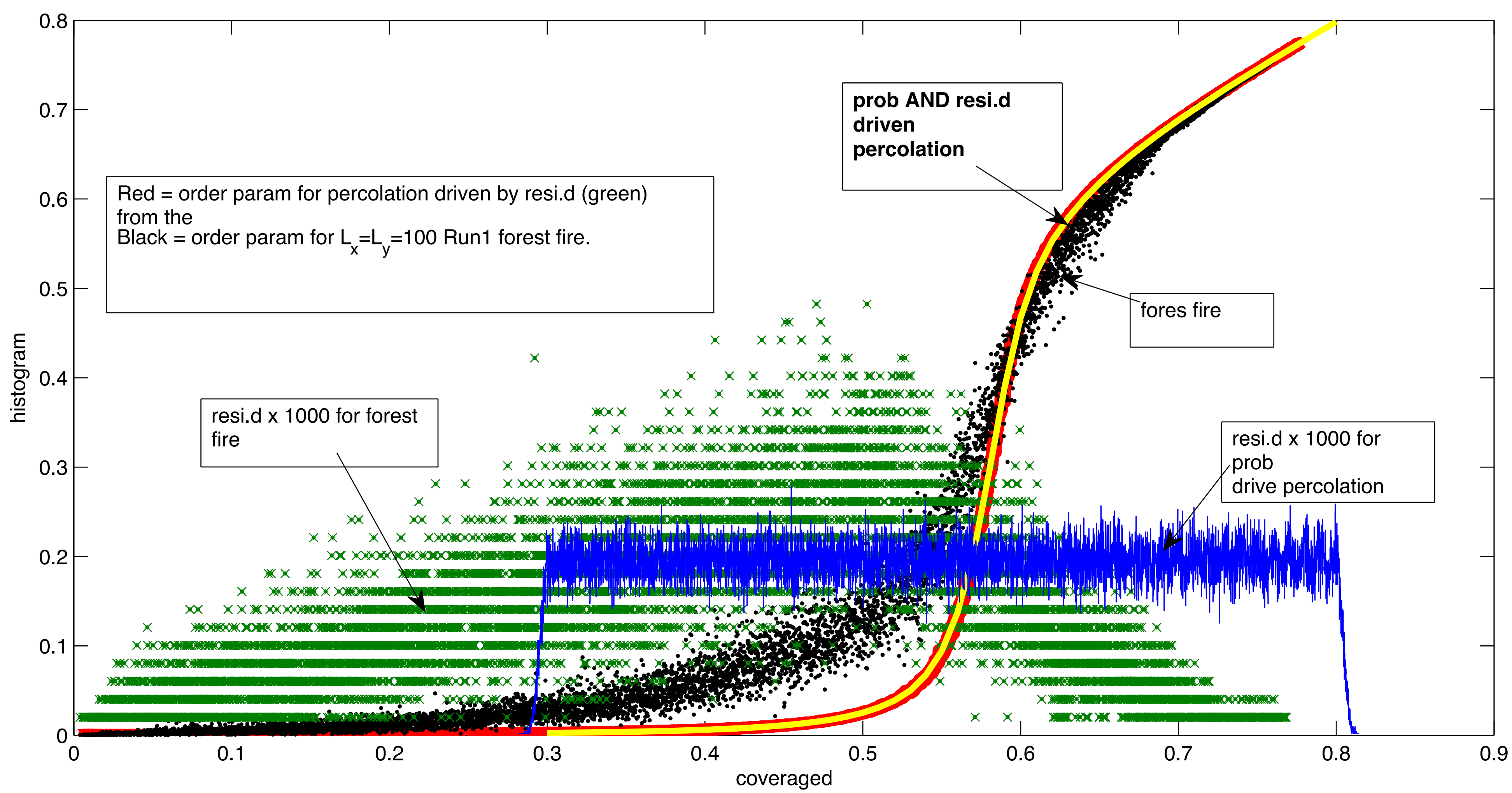
- Consider a square lattice
- Plant trees with probability p
- Ignite trees with probability $f << p$
- Tree neighbour to a burning tree catches fire

- If one tree in a cluster is on fire the entire cluster will burn down.
- Related to percolation

Investigate the statistics of cluster sizes in percolation and in the forest fire model

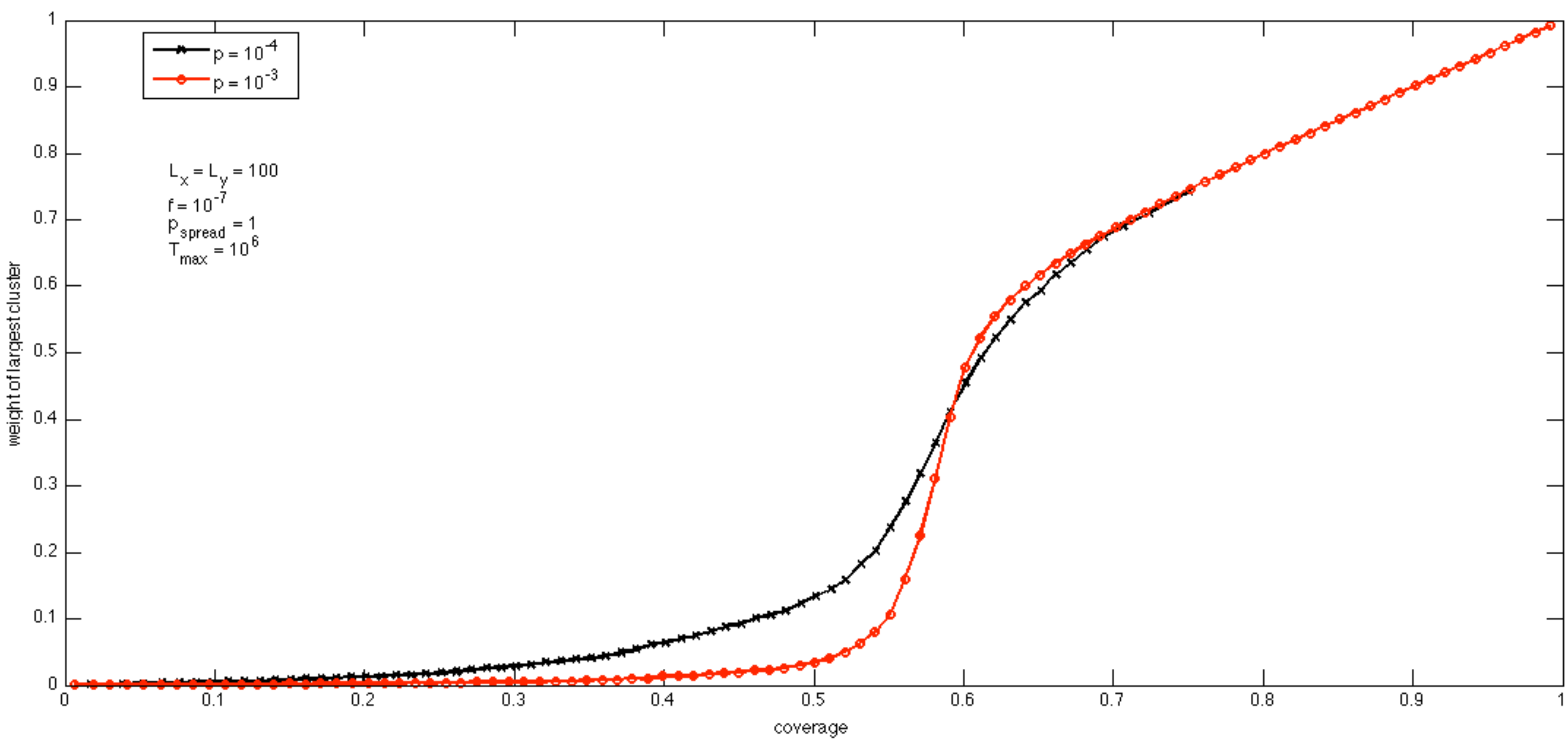

max cluster
susceptibility x 15
residence time
Lx=100 Ly = 100
p = 10^-4, f = 10^-7
p_soread = 1
T_max = 10^6
Measure = 5000
histogram
occupancy
x 10^-3
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
1
0.5
0

Complexity & Networks



Forest fire

Order parameter plot for different growth probabilities



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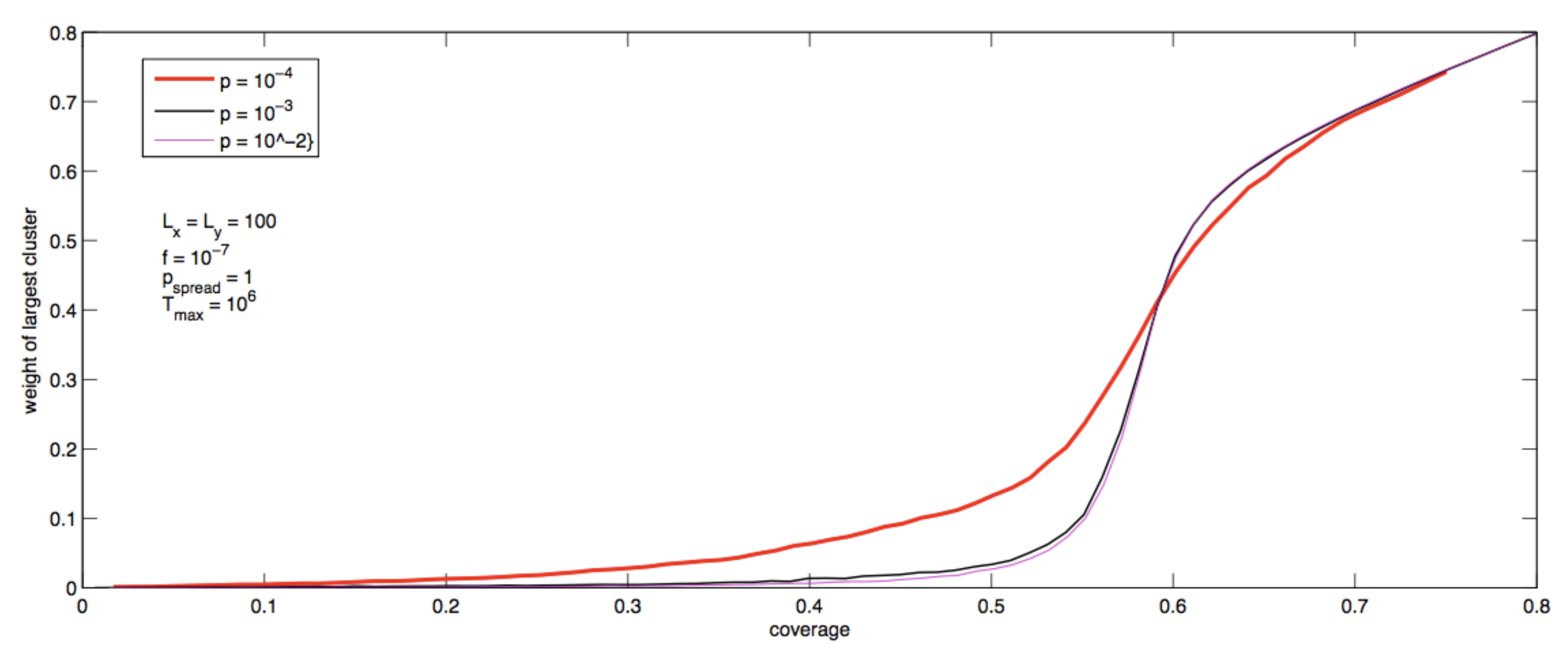

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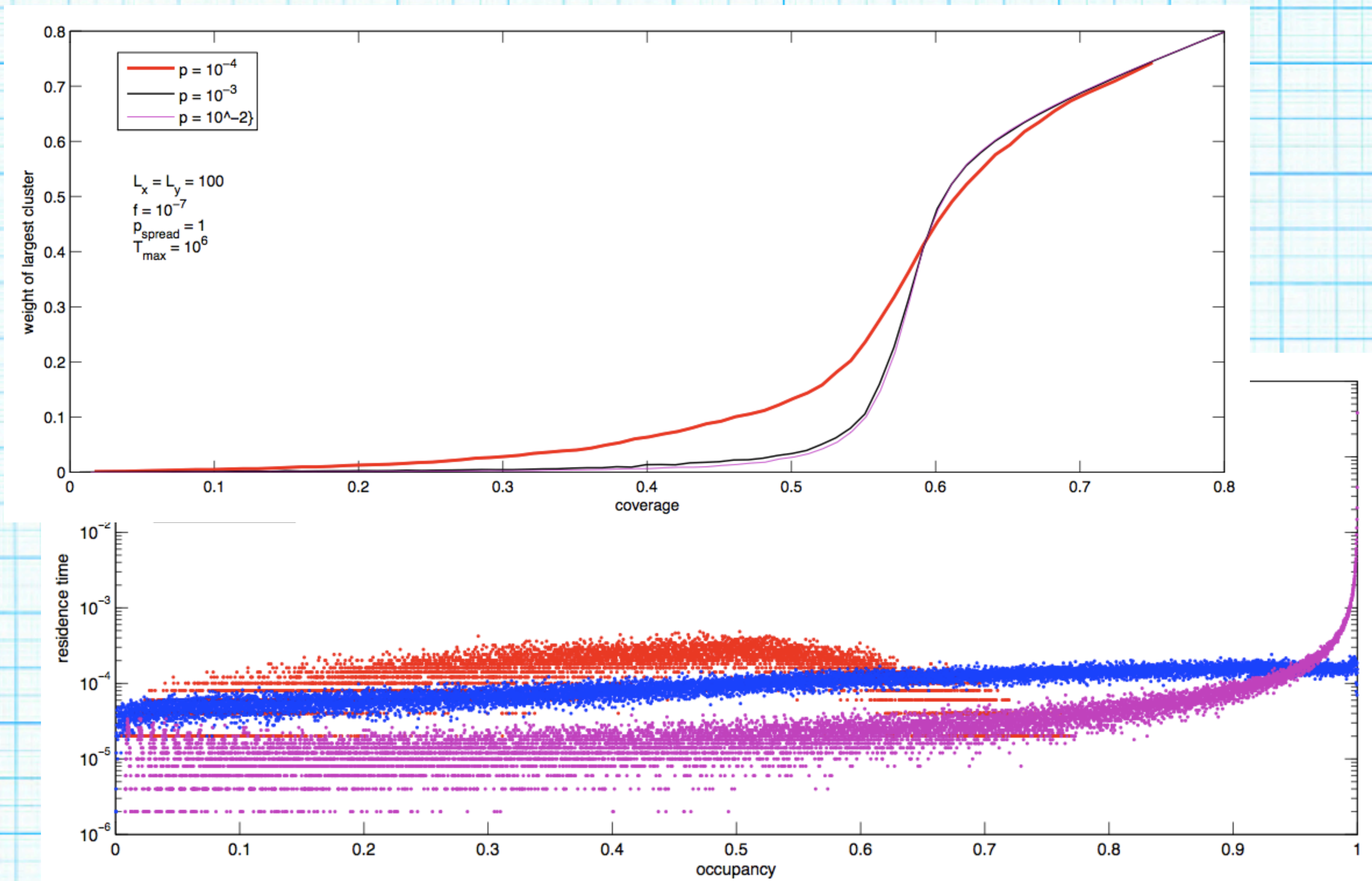
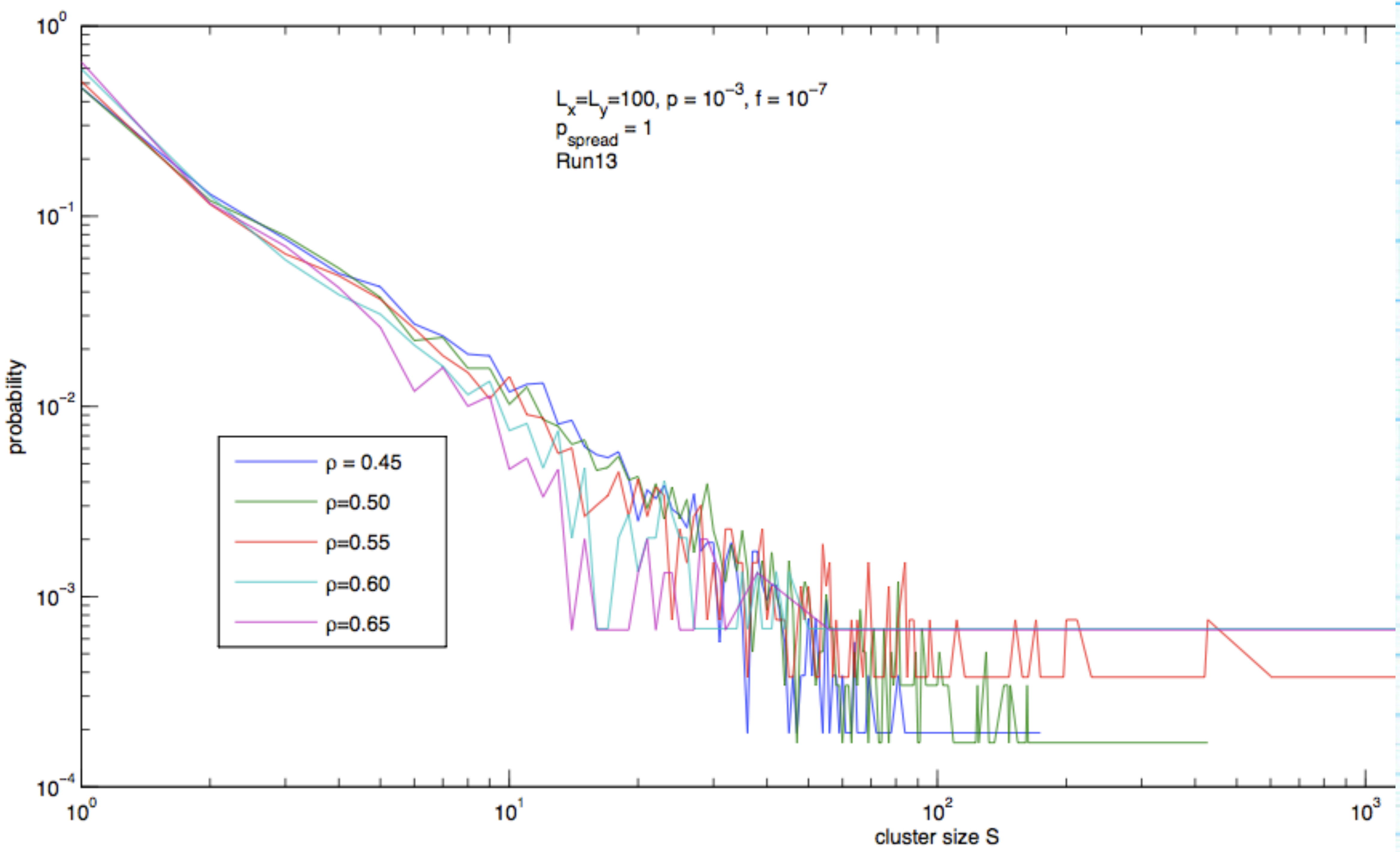
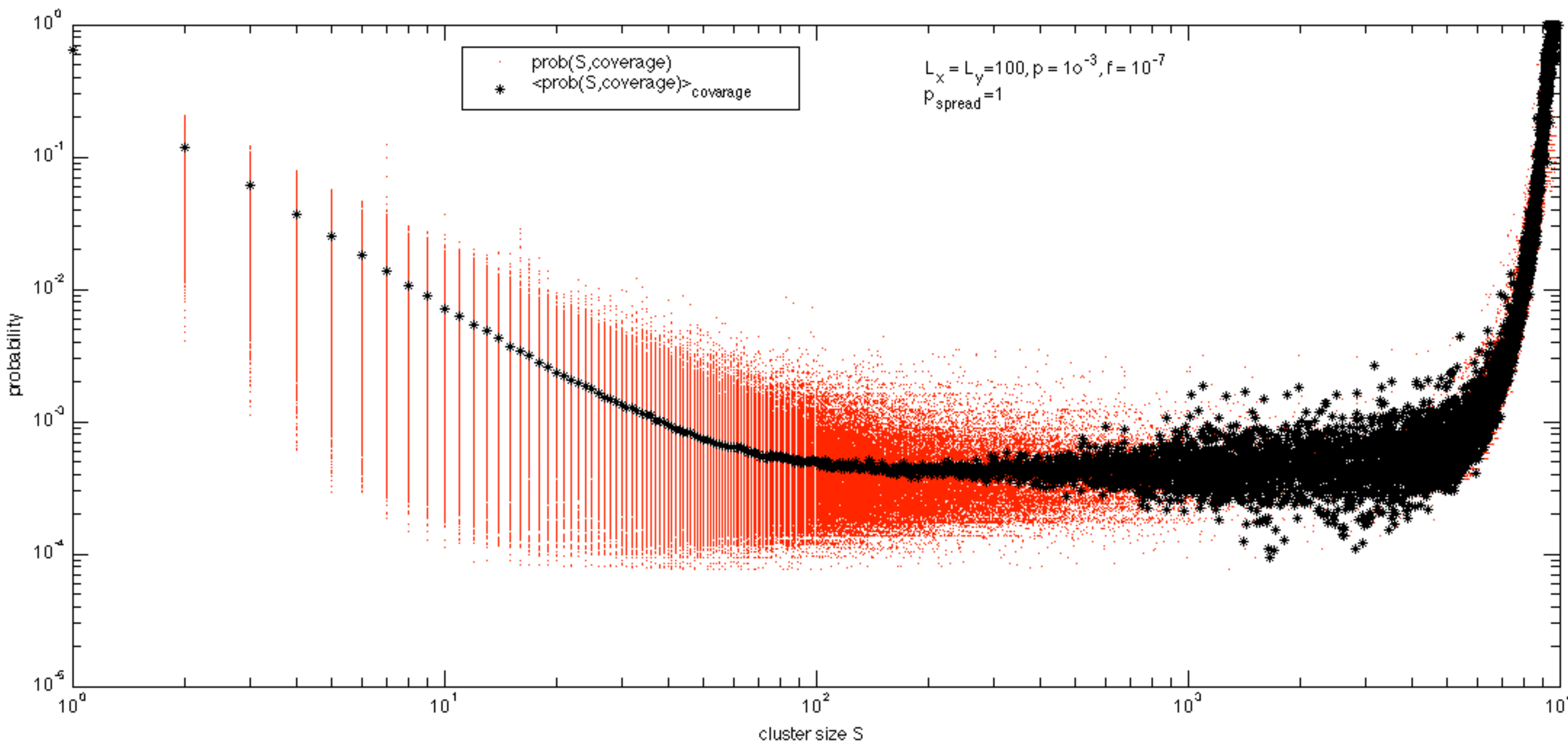


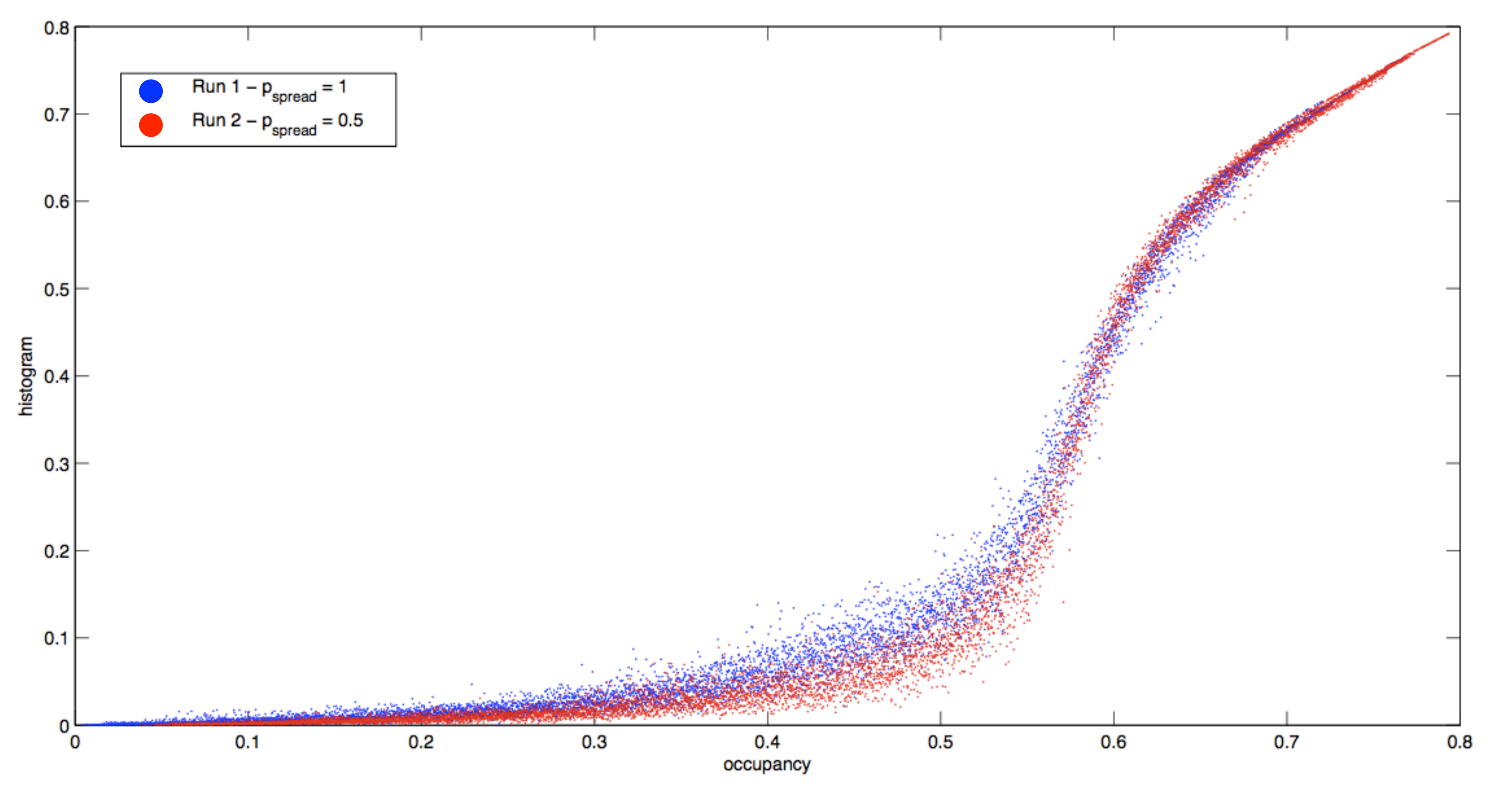
Cluster size histogram for different coverages



Cluster size histogram averaged over coverages



Effect of the coupling between the order parameter and the dynamics

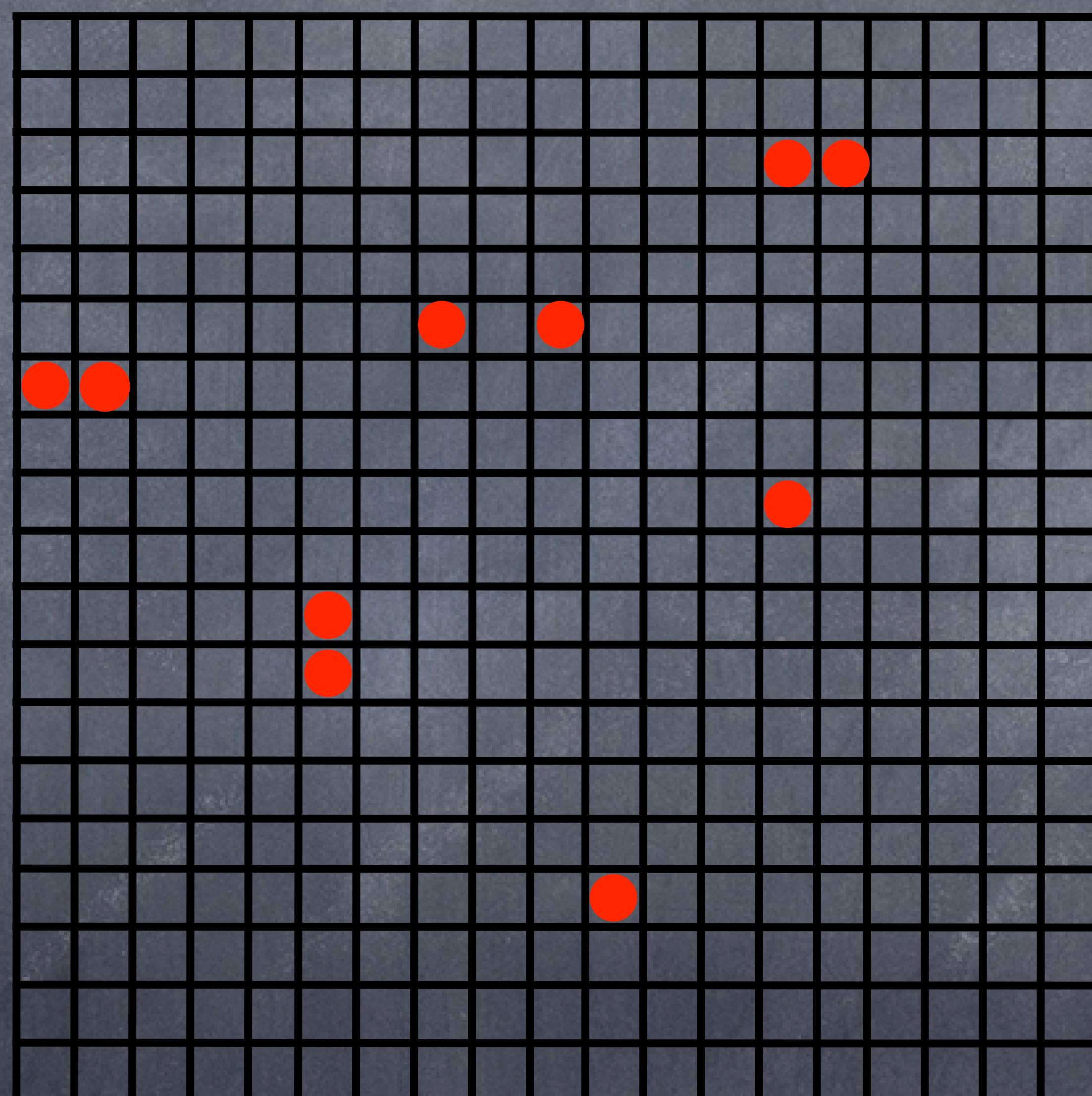


Lattice gas

Deterministic lattice gas

Periodic boundary condition to avoid surface effects

$$N(t)$$



Density dependence

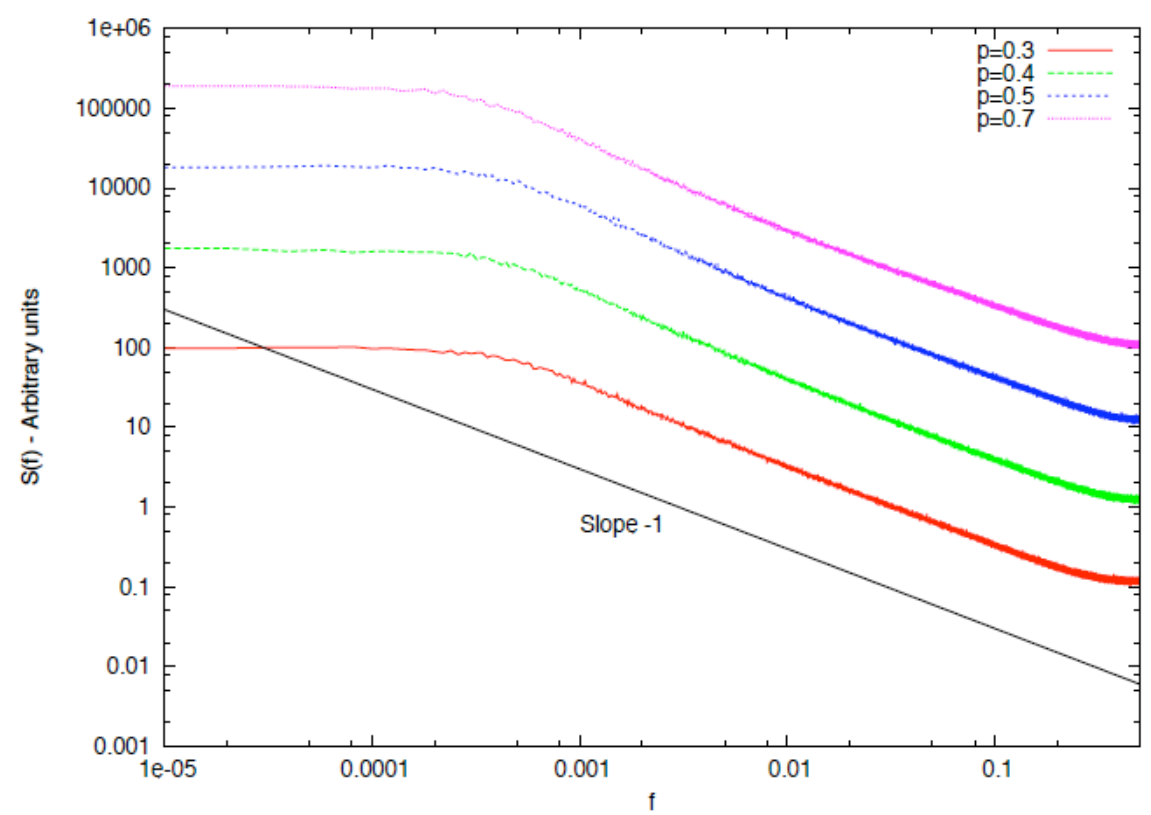


FIG. 12: Scaling behaviour of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles $N(t)$ in the dDLG for different boundary drives p and particles densities ρ. $S(f)$ has been multiplied by different constants for different drives p to visualise the scaling exponents properly. With small lattice sizes one observes scaling with $\beta = 1$. Lattice size $L = 64$.

From Master thesis
Andrea Giomette

Individual particles behave as random walkers at times later than about 100 time steps

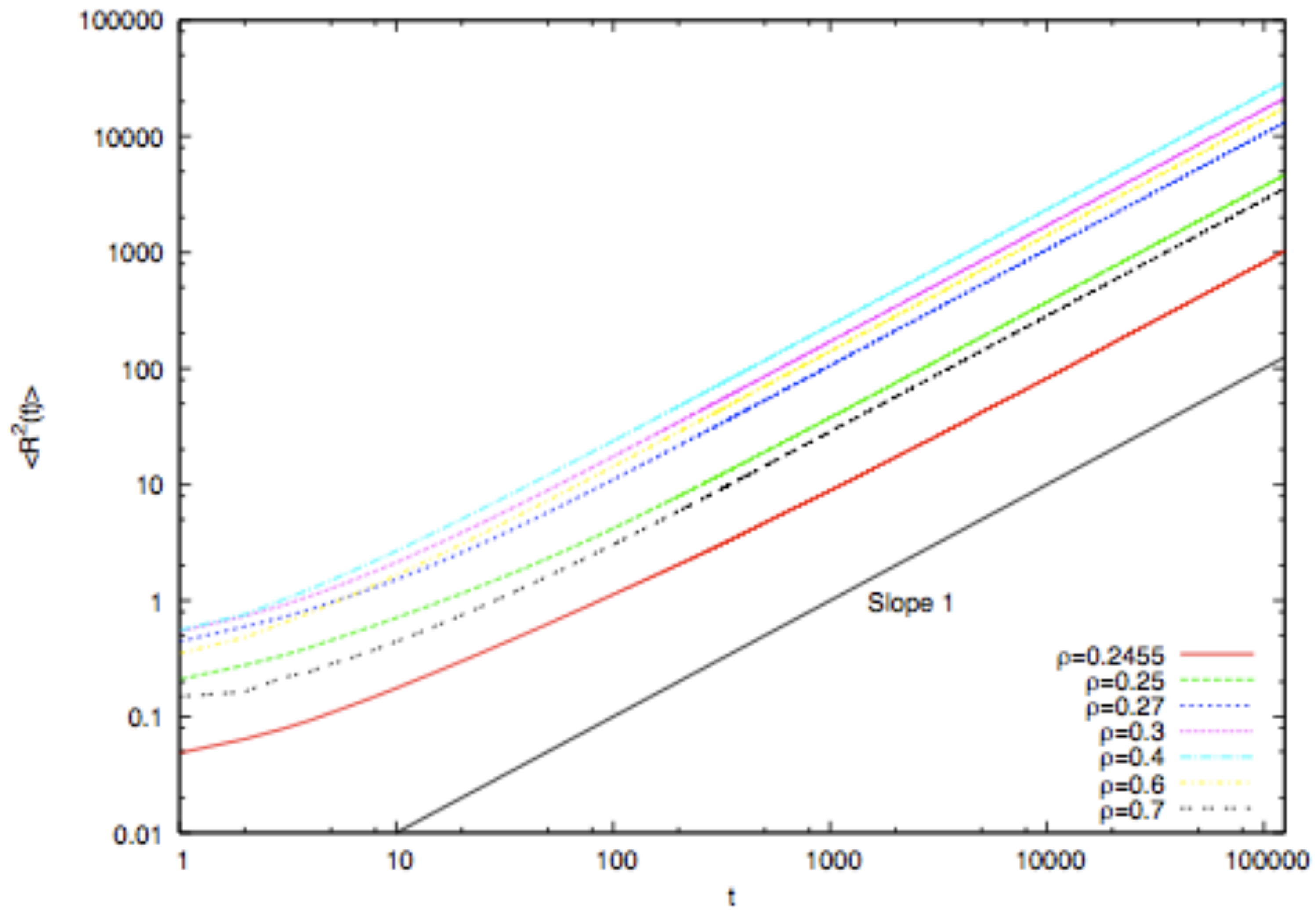


Figure 2.1: pDLG: Mean square displacement $\langle R^2(t) \rangle$ versus time t for different particles densities. Individual particles asymptotically experience ordinary random walks. $L = 250$.

Even bigger systems

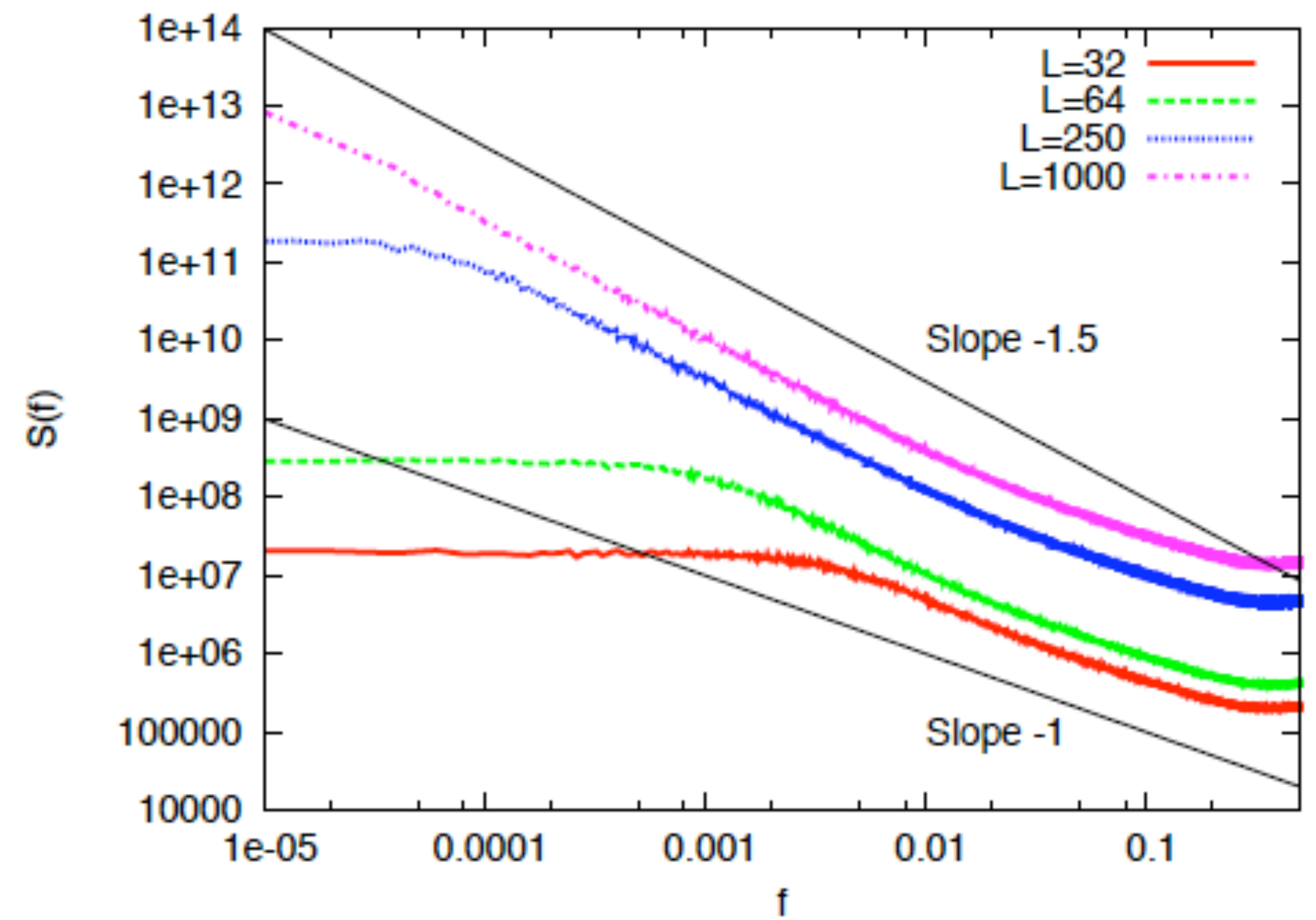


FIG. 1: Scaling behavior of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles $N(t)$ in the DLG for increasing linear sizes of the lattice L. A crossover from $\mu = 1$ for small L to $\mu = 1.5$ for large L is observed. Particle density $\rho = 0.5$.

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Andrea Giomette

Even bigger systems
- and density dependence

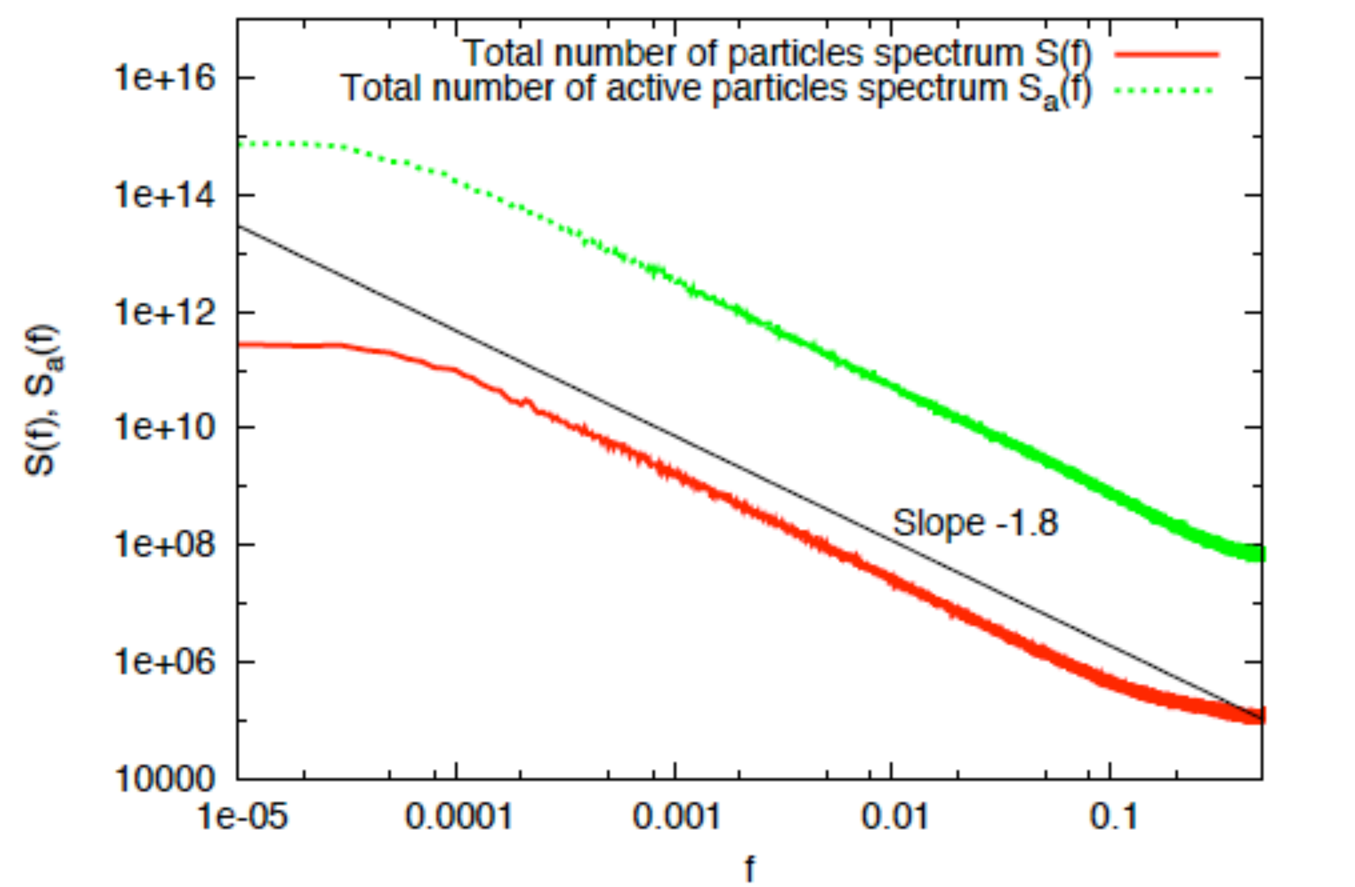

FIG. 10: DLG: Scaling behavior of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles in the box $N(t)$ and of the spectrum $S_a(f)$ of the total number of active particles near the critical density $\rho = 0.24506 \simeq \rho_c$. The spectra have been multiplied by arbitrary factors to visualize the scaling exponent properly. Lattice linear size $L = 1000$.

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Larger systems

> at higher densities => μ = 3/2

> at low density => μ = 1.8

Explanation The β = 3/2 at high density

As the system increases a bulk noise term is generated

$$\frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t) \quad \Rightarrow \quad \frac{\partial n(\mathbf{x},t)}{\partial t} = \gamma \nabla^2 n(\mathbf{x},t) + \nabla \cdot \vec{\chi}(\mathbf{x},t)$$

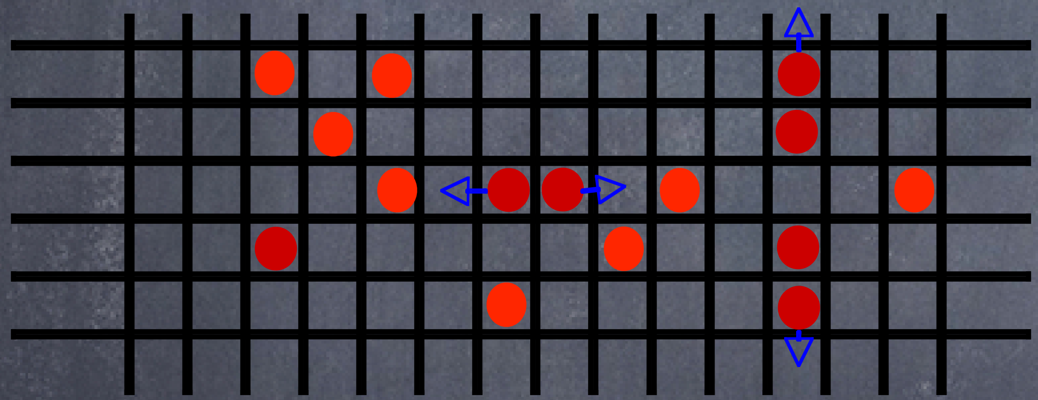
$$\mu = 3/2$$

44

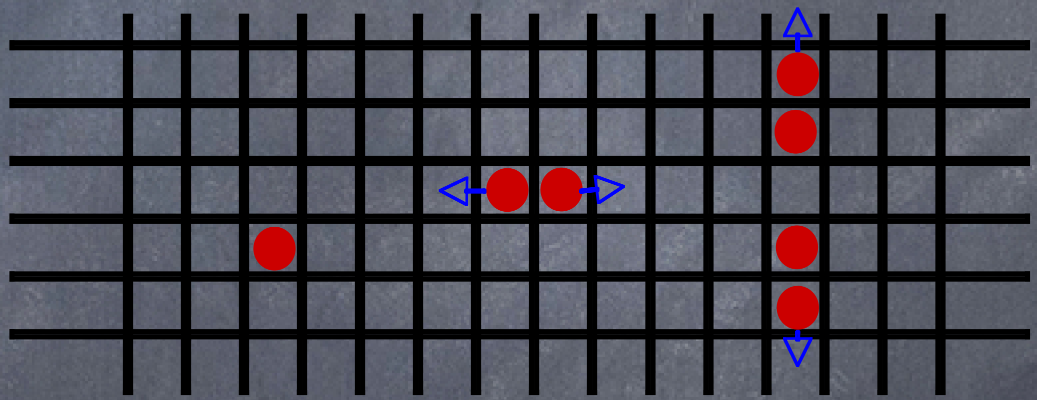
Explanation The β = 1.8 at low density

The absorbing state phase transition:

> as density decreases motion will stop

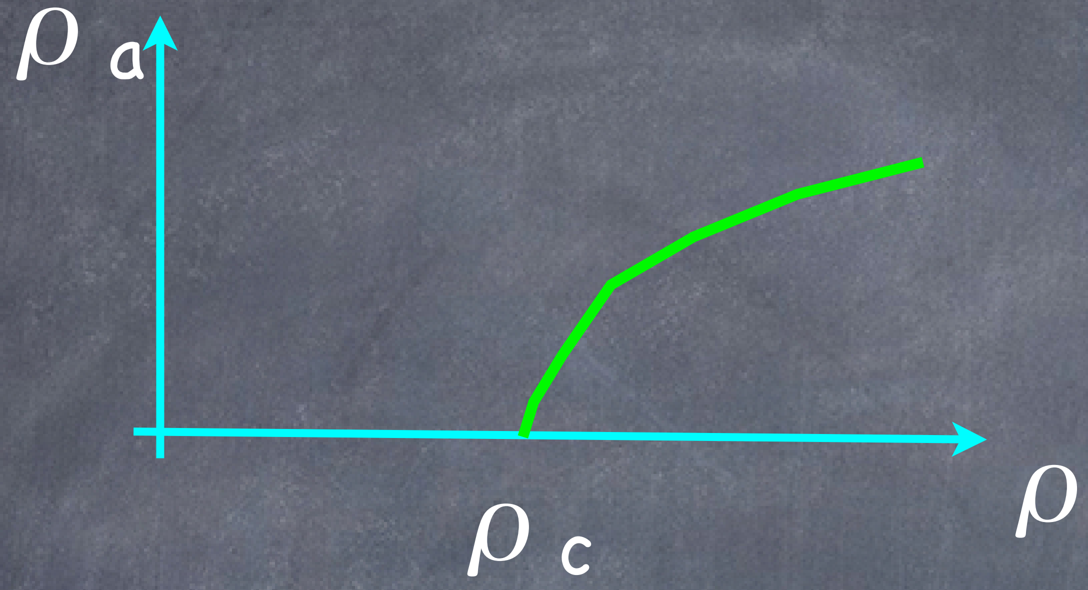


High density



Low density

The absorbing state phase transition



Density of active particles $\rho_a \sim (\rho - \rho_c)^{\beta}$

Survival probability $P_\infty \sim (\rho - \rho_c)^{\beta'}$

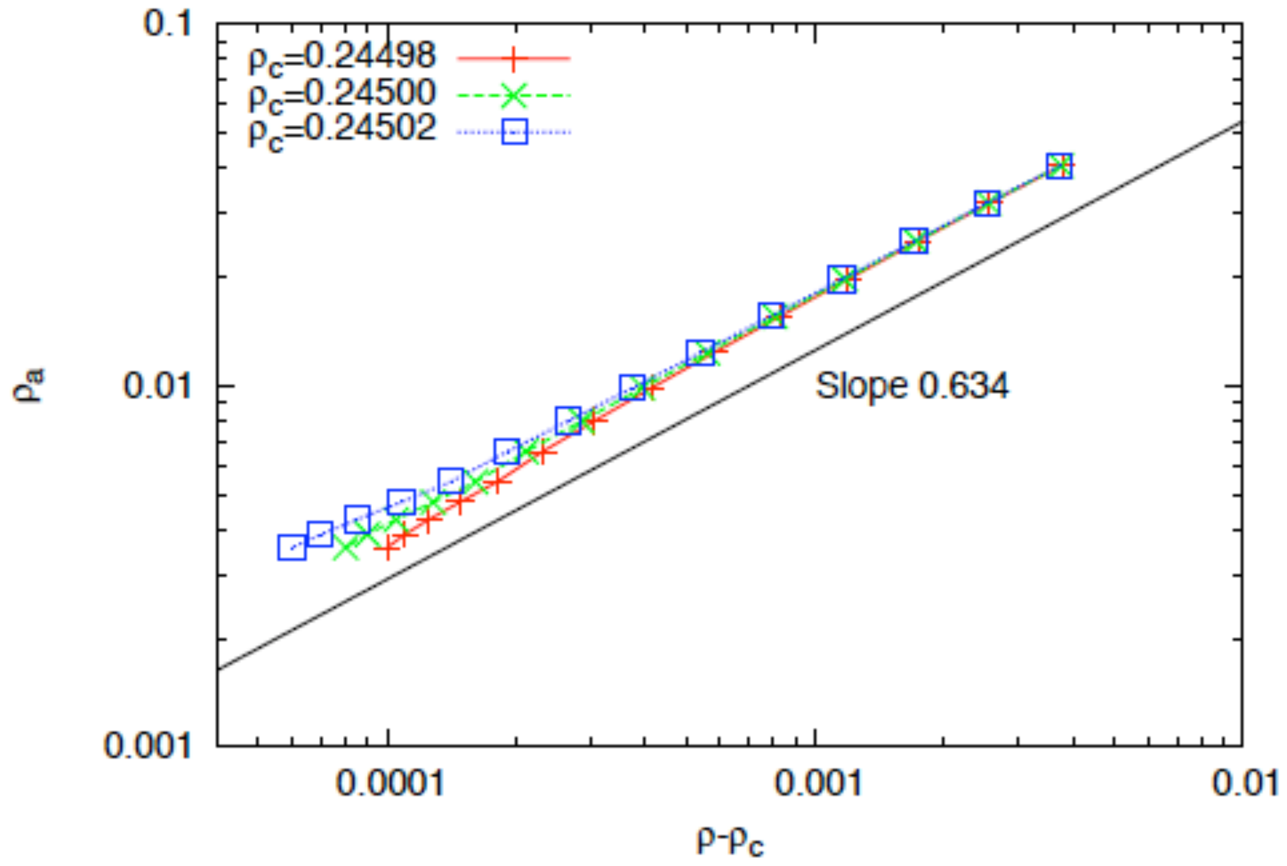


FIG. 3: Density of active sites ρ_a as a function of $\delta\rho = \rho - \rho_c$. The determination of the critical density ρ_c is obtained varying its value until data points are aligned on a straight line in a log-log plot. Lattice linear size $L = 1000$. Error bars are smaller than symbols.

$\beta = 0.634$

Study all the critical exponents

Determine universality class

	β	$\nu_\perp$	$\nu_\parallel$	σ
Manna	0.639(9)	0.799(14)	1.225(29)	2.229(32)
DLG	0.634(2)	0.83(5)	1.2(1)	2.19(1)

	γ'	γ	α	z
Manna	0.367(19)	1.590(33)	0.419(15)	1.533(24)
DLG	0.37(1)	1.54(1)	0.41(1)	1.5(1)

TABLE I: The measured critical exponents for the DLG and the corresponding critical exponents for the Manna universality class in $d = 2$[12].

Conclude:

Deterministic Lattice Gas belongs to the Manna universality class.

Hitherto unexpected because because all other members are stochastic models.

Back to power spectrum

$$\mu = 1 + \frac{1}{z}(2 - \frac{\beta}{\nu_\perp})$$

$$\begin{aligned} \beta &= 0.634 \\ z &= 1.5 \\ \nu &= \perp = 0.83 \end{aligned}$$

$$\mu = 1.78(2)$$

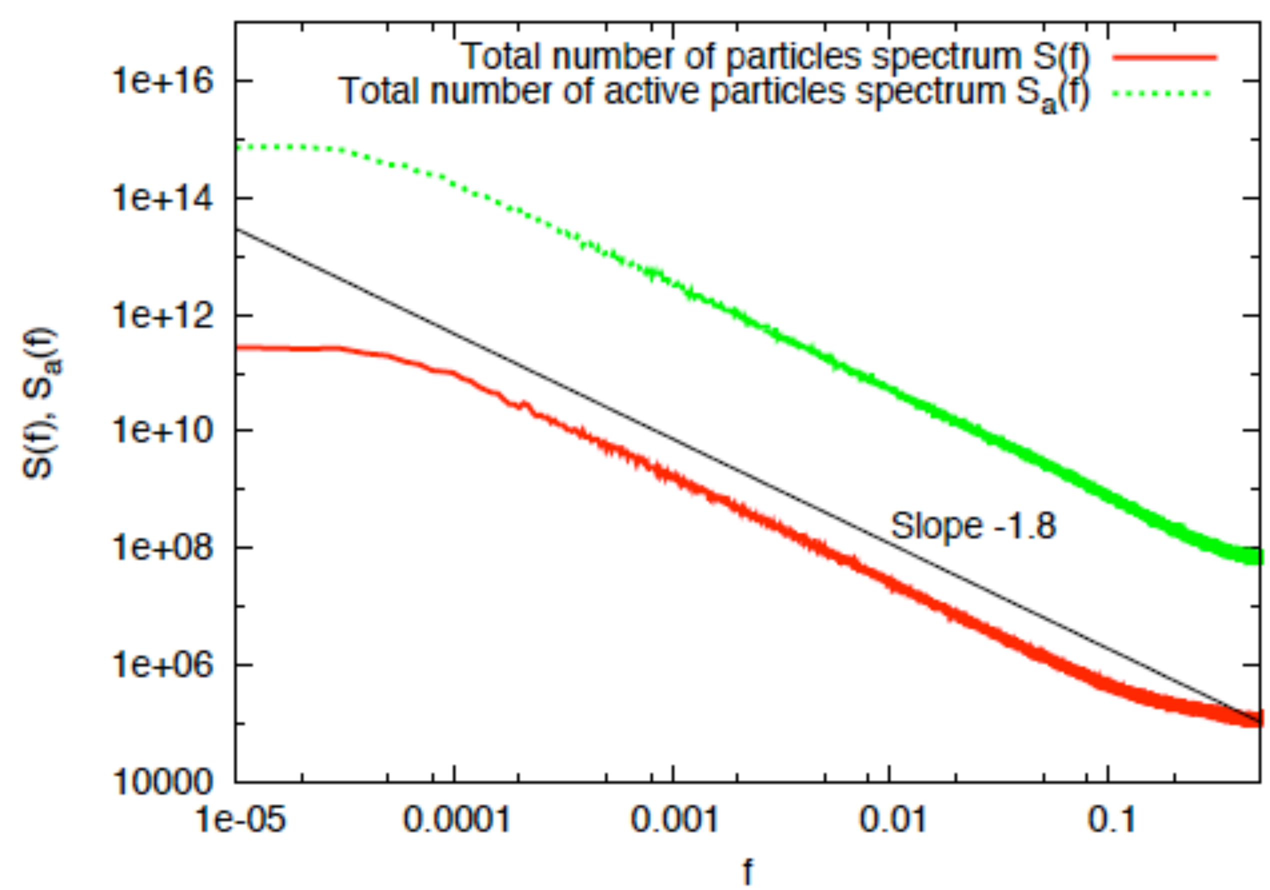


FIG. 10: DLG: Scaling behavior of the spectrum $S(f) \propto f^{-\mu}$ of the total number of particles in the box $N(t)$ and of the spectrum $S_a(f)$ of the total number of active particles near the critical density $\rho = 0.24506 \simeq \rho_c$. The spectra have been multiplied by arbitrary factors to visualize the scaling exponent properly. Lattice linear size $L = 1000$.

$\rho \approx \rho_a$ near transition

Summary of lattice gas behaviour:

> phenomenology well described by Langevin Eq. with bulk noise at elevated densities

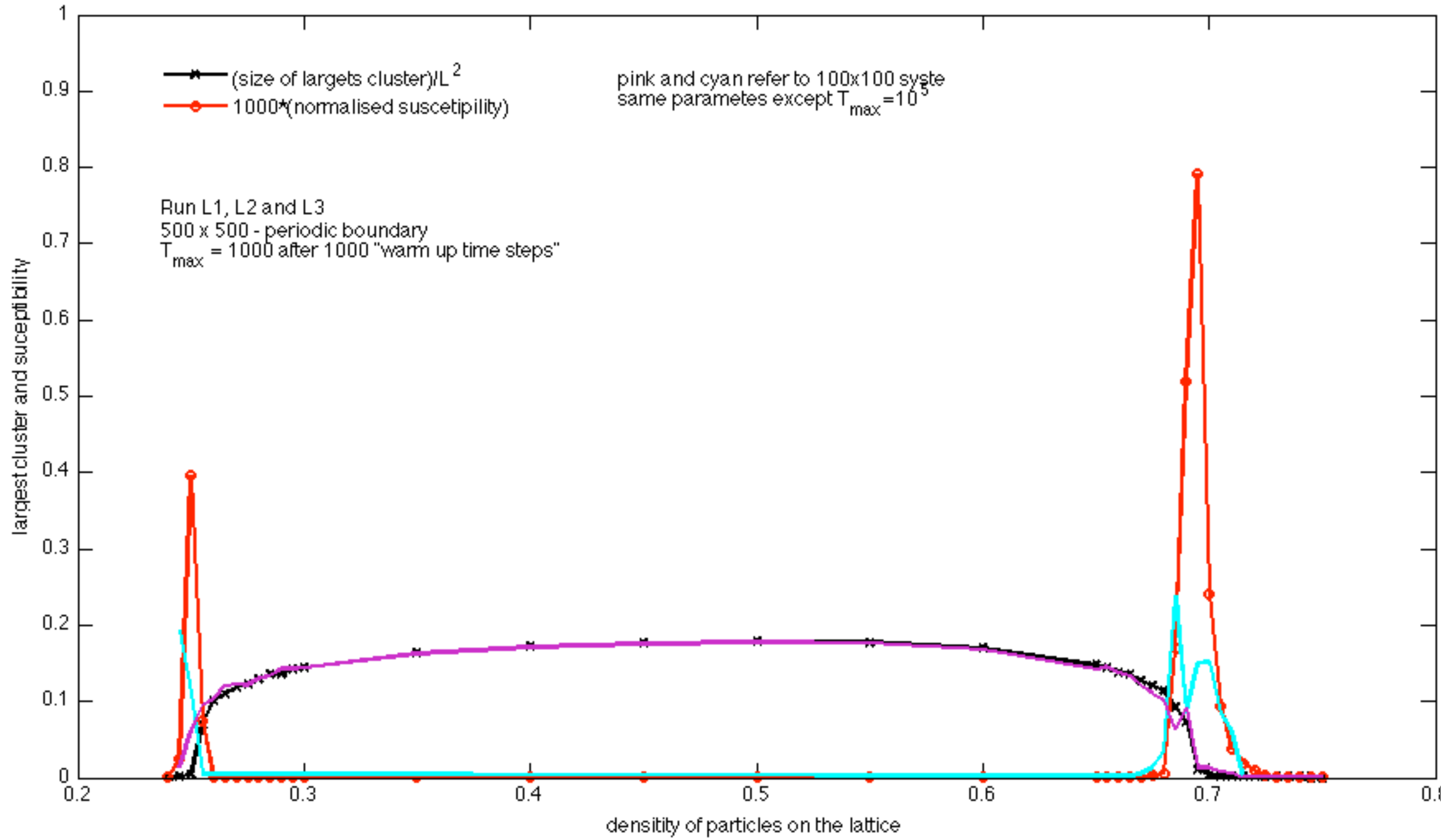
> at low densities near the frozen state critical properties describes the fluctuations

The morale

- From study of power laws the lattice gas appeared to be critical for all densities.
- Combination of theory and simulations established proper critical behaviour only at the absorbing state phase transition
- Take a close look at the fluctuations

Cluster analysis of sites involved in dissipation

Cluster analysis of sites involved in dissipation



Conclusion

- Brain and rain certainly not in a critical state
  - but certainly wanders around near one
- Identify the “order parameter”
  - and study the fluctuations of it
    = susceptibility

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Thank you