Any answers?

Self Organised Criticality in the third decade after BTW

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Bern (ISSI meeting), 15 October 2012

Outline

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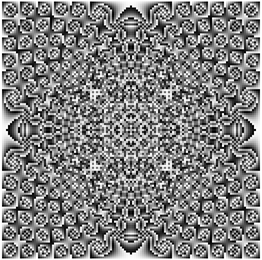
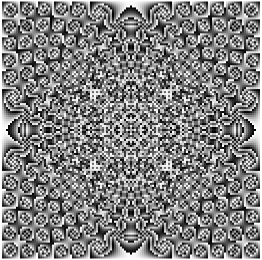
Prelude: The physics of fractals



Question: Where does scale invariant behaviour in nature come from?

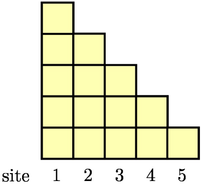
Answer: It is due to a phase transition, self-organised to the critical point.

Prelude: The physics of fractals



- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- $1/f$ noise “everywhere” (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where’s the Physics?*
- Bak, Tang and Wiesenfeld, 1987: *Self-Organized Criticality: An Explanation of $1/f$ Noise* (later: The physics of fractals)

The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

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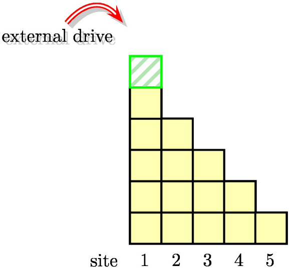

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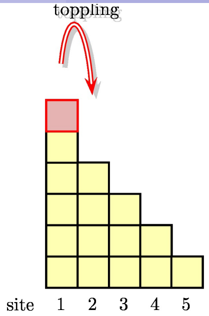

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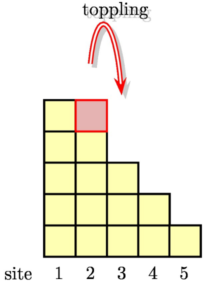

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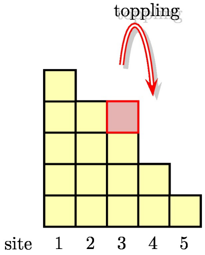

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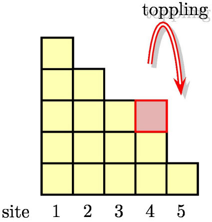

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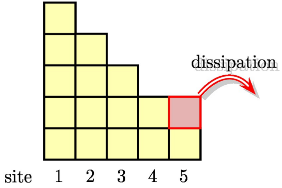

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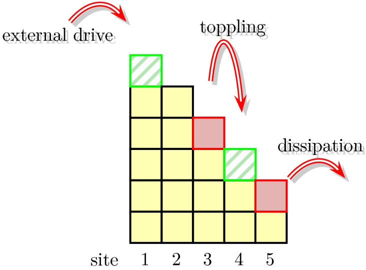

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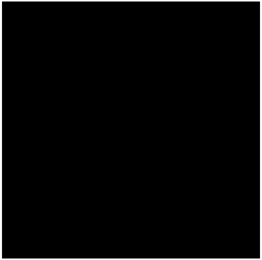

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What is Self-Organised Criticality (SOC)?

Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.
- **Scale invariance in space and time: Emergence! Universality!**

Universal (?) exponents τ and D

$$\mathcal{P}(s;L) = a s^{-\tau} \mathcal{G}\left(\frac{s}{bL^{D}}\right)$$

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Why is SOC important?

SOC: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Emergence!

- Explanation of emergent,
- . . . cooperative,
- . . . long time and length scale
- . . . phenomena,
- . . . as signalled by power laws.

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Why is SOC important?

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Universality!

- Understanding and classifying natural phenomena
- ...using *Micky Mouse Models*
- ...on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?

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$1/f$ noise — a red herring? I

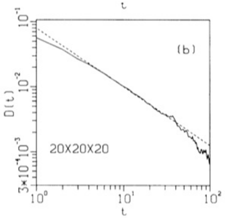

FIG. 3. Distribution of lifetimes corresponding to Fig. 2. (a) For the 50×50 array, the slope $\alpha\approx0.42$, yielding a "$1/f$" noise spectrum $f^{-1.58}$; (b) $20\times20\times20$ array, $\alpha\approx0.90$, yielding an $f^{-1.1}$ spectrum

From: Bak, Tang, Wiesenfeld, 1987

- Power spectrum $P(f)\propto 1/f$, thus correlation function (via Wiener Khinchin) decays "very slowly".

$1/f$ noise — a red herring? II

- Dimensional analysis:

$$\int \mathrm{d}f\, 1/f^{\alpha} e^{-2\pi\imath f t} = \ldots \propto t^{\alpha-1} = \text{const}$$

- $1/f$ noise suggests long time correlations
- Initially, SOC was intended an explanation of $1/f$ noise.
- Initially the BTW model was thought to display $1/f$ noise.
- Jensen, Christensen and Fogedby: “Not quite.”
- Today: Reduced interest in $1/f$.
- Today: Power laws in other observables.

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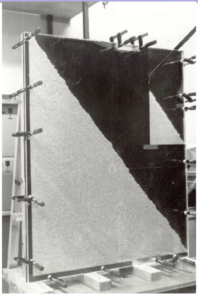
Experiments:

Granular media, superconductors, rain. . .



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

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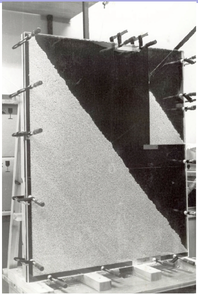

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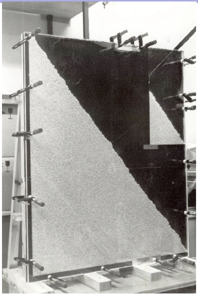

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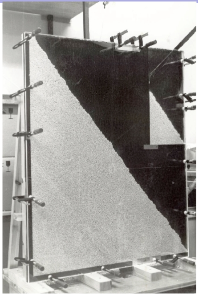

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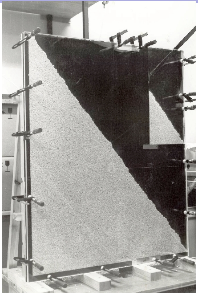

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Outline

More models

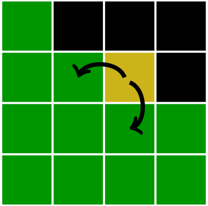
- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models. . .

More models

The failure of SOC?

- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, $\alpha \approx 0.05$ (localisation), $\alpha = 0.22$ (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]
- Directed Models: Exactly solvable (lack of correlations)

The Bak-Chen-Tang Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

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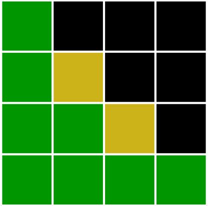

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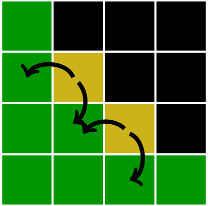

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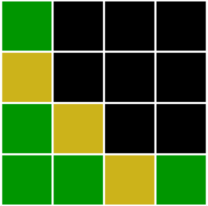

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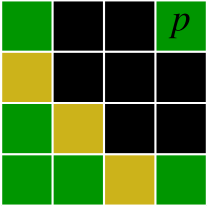

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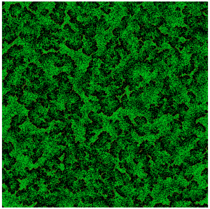

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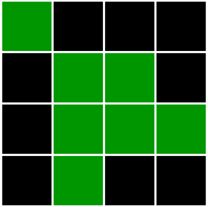

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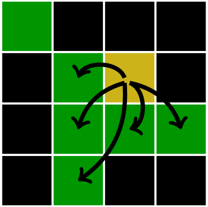

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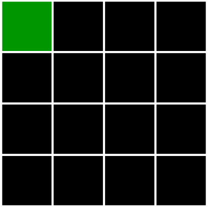

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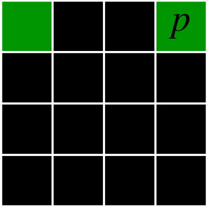

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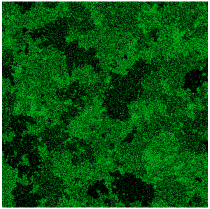

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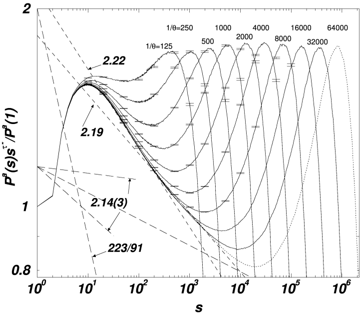
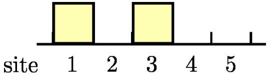
Lack of scaling



- Finite size not the only scale.
- Scale invariance possible only in the limit of $\theta \to \infty$.
- Lower cutoff moves as well.

Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

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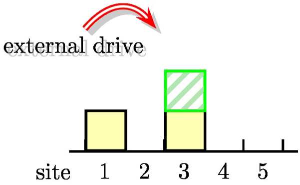

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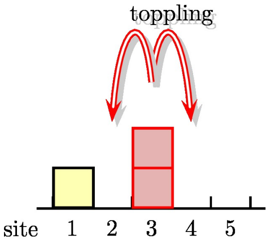

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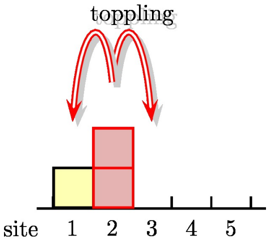

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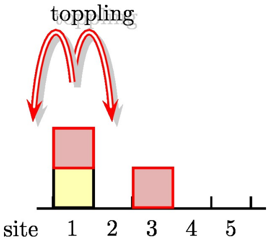

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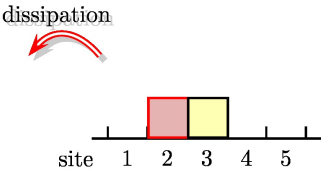

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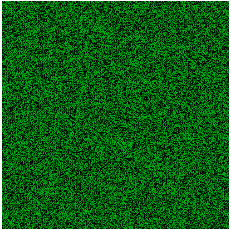

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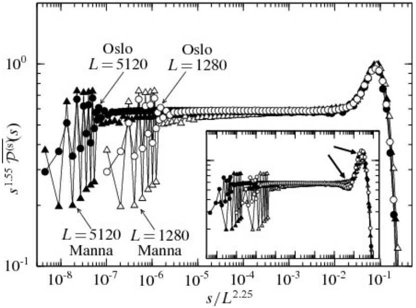
- Robust, solid, universal, reproducible.
- Defines a universality class.

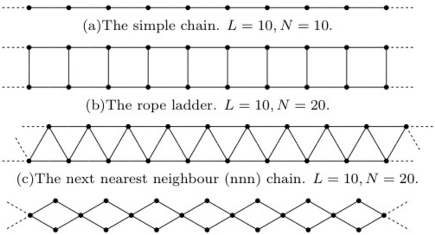
Collapse with Oslo



The Manna Model is in the same universality class as the Oslo model.

Manna on different lattices

One and two dimensions



(a)The simple chain. $L = 10, N = 10$.

(b)The rope ladder. $L = 10, N = 20$.

(c)The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

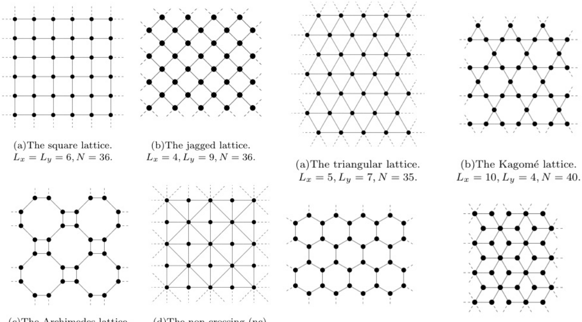
(d)The Futatsubishi lattice. $L = 7, N = 22$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

One and two dimensions



(a)The square lattice. $L_x = L_y = 6, N = 36$.

(b)The jagged lattice. $L_x = 4, L_y = 9, N = 36$.

(c)The Archimedes lattice. $L_x = 8, L_y = 4, N = 32$.

(d)The non-crossing (nc) diagonal square lattice. $L_x = L_y = 5, N = 25$.

(a)The triangular lattice. $L_x = 5, L_y = 7, N = 35$.

(b)The Kagomé lattice. $L_x = 10, L_y = 4, N = 40$.

(c)The honeycomb lattice. $L_x = 9, L_y = 4, N = 36$.

(d)The Mitsubishi lattice. $L_x = 5, L_y = 7, N = 35$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

One and two dimensions

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
simple chain	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
rope ladder	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
nnn chain	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
Futatsubishi	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
square	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
jagged	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
Archimedes	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
nc diagonal square	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
triangular	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
Kagomé	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
honeycomb	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
Mitsubishi	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

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Manna on different lattices

Three dimensions

Lattice	$\overline{q}$	$\overline{q^{(v)}}$	$\langle z \rangle$	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SC	6	1	[0.622325(1)]	3.38(2)	1.408(3)	1.779(7)	1.784(9)	3.04(5)	1.45(4)	2.0057(5)	1.38(2)	1.395(16)	1.36(13)
BCC	8	4	[0.600620(2)]	3.36(2)	1.404(4)	1.777(8)	1.78(1)	2.99(2)	1.444(18)	2.0030(5)	1.36(2)	1.390(19)	1.33(6)
BCCN	14	5	[0.581502(1)]	3.38(3)	1.408(4)	1.776(9)	1.783(11)	3.01(3)	1.44(3)	2.0041(6)	1.38(3)	1.39(2)	1.32(7)
FCC	12	4	[0.589187(3)]	3.35(4)	1.402(8)	1.765(16)	1.78(2)	3.1(2)	1.48(14)	2.0035(11)	1.35(4)	1.37(4)	1.5(5)
FCCN	18	5	[0.566307(3)]	3.38(4)	1.408(7)	1.781(14)	1.787(18)	3.00(4)	1.44(3)	2.0051(8)	1.38(4)	1.40(3)	1.32(9)
Overall				3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	2.0042(3)		1.380(13)	

From: Huynh, G P, 2012

The Manna Model has been investigated numerically in great detail.

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Outline

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Theoretical tools in SOC

- (Extensive) numerics (BTW, FFM, BS, Manna, Oslo).
- Analytical tools:
  - Exact solutions (so far: directed models only).
  - Mappings to known (understood?) phenomena.
  - Growth processes and field theories.

It's all about power laws. Why?

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Power law correlation function

> *"Why is a power law any different from any other functional dependence? What is the physical significance of scaling?"*

Full scaling[1] — pure power law: No scale from within.
Example:

- Exponential correlations, $C(r) = \exp(-x/\xi)$. Correlation length[2] = distance over which correlations decay by e^{-1}.

$$C(r+\xi) = C(r)/e$$

- Power law, $C(r) = ar^{-2}$: Correlations decay by the same factor at every multiple:

$$C(r\sqrt{e}) = C(r)/e$$

[1] As opposed to finite size scaling with intermediate power law scaling.
[2] In general, this holds only asymptotically.

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Power law PDFs

Simple scaling:

$$\mathcal{P}(E) = a^{\tau-1} E^{-\tau} \mathcal{G}\left(\frac{E}{E_c}\right) \quad \text{for } E \gg E_0$$

rather than[1]

$$\mathcal{P}(E) = a^{-1} e^{-E/a}$$

Other scales are present without destroying the scaling.

There is an *arbitrarily wide*, intermediate range of power law scaling.

Different physics kicks in below and above a certain scales. In between: The same physics throughout.

[1] Below can be cast in the form above with $\tau = 1$, but then macro=micro, $E_c = a$.

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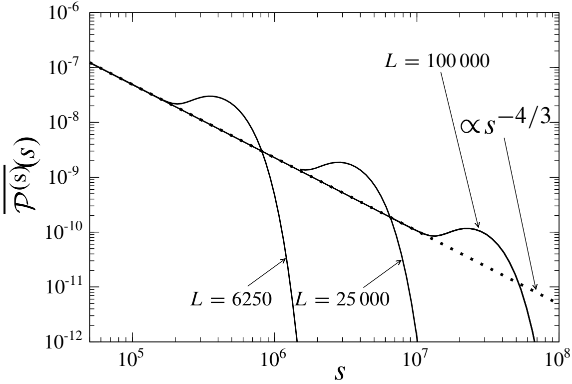
Power law PDFs

Other scales are present without destroying the scaling.

There is an *arbitrarily wide*, intermediate range of power law scaling.



Power laws frequently don't apply.

> *"Nature is different and more complicated."*
> *(e.g. Avnir, Biham, Lidar, Malcai, 1998)*

Perfect power laws are much less common than alleged.

A year in the lab is often not enough to extract the allegedly *ubiquitous* power law.

Nature is full of *dirty* power laws, "almost scaling".

Problem: Publication bias and self-selection.

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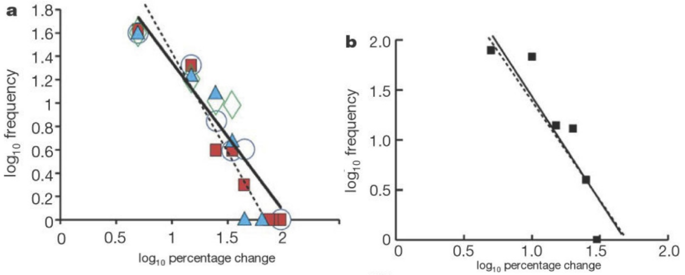
Power laws frequently don't apply.

"Nature is different and more complicated."
(e.g. Avnir, Biham, Lidar, Malcai, 1998)

Nature is full of *dirty* power laws, "almost scaling".



(Freckleton and Sutherland, 2001)

Problem: Publication bias and self-selection.

Power laws frequently don't apply.

Power laws are misunderstood!

Powerlaws do NOT indicate unpredictability and/or optimisation

- Predictability: Power law correlated events are predictable (Gutenberg and Richter law).
- Optimisation: Large susceptibility is an optimum of what? (HOT? COLD? TEPID?)

Why get excited about power laws?

Narrative: **If** power laws are observed in a PDF (or other observable) on an arbitrarily large but intermediate range:

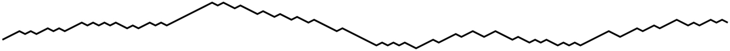
- …they may be caused by power law correlations (but power law PDFs are not indicative of power law correlations and vice versa).
- …they indicate the absence of an intrinsic scale.
- …they are the signature of emergence, collective behaviour, "more is different" (Anderson, 1972), extreme events(?).
- Exponents identify universality classes.
- Exponents characterise observables ("summary" of a PDF).

Power laws are not an end in itself.

Link to growth phenomena

Generic scale invariance

Stochastic evolution of sandpile surface.



$$\partial_t \phi(\mathbf{r}, t) = (\nu_{\|} \partial_{\|}^2 + \nu_{\perp} \partial_{\perp}^2) \phi + \eta(\mathbf{r}, t)$$

- *Generic* scale invariance (Hwa and Kardar, 1989, and Grinstein, Lee and Sachdev 1990)
- No mass term $-\epsilon\phi$ on the right $\longrightarrow$ conservative dynamics (finiteness generates ϵ).
- Anisotropy (boundaries?) required in the presence of conserved noise.
- Non-trivial exponents in the presence of non-linearities and non-conserved noise.

Effect of a mass term

Mass term

$$\partial_t \phi = \nu \nabla^2 \phi - \epsilon \phi + \ldots + \eta$$

represents disspation

$$\partial_t \int_V \mathrm{d}^d x \, \phi = \text{surface terms} - \epsilon \int_V \mathrm{d}^d x \, \phi$$

and correlation length

$$\phi = \ldots e^{-|x|\sqrt{\epsilon/\nu}} .$$

But: How can a renormalised $\epsilon = 0$ be maintained without trivialising the phenomenon?

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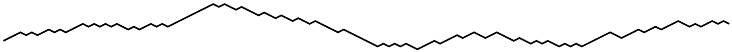
Field theories for Manna and Oslo

Number of charges interpreted as an interface.



- **Manna model** has a (weird!) Langevin equation.
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!
- Field theories for both still unclear.
- Mechanism of self-organisation still unclear.
- Link to known universality classes.
- Link to **directed percolation**?

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The Absorbing State Mechanism

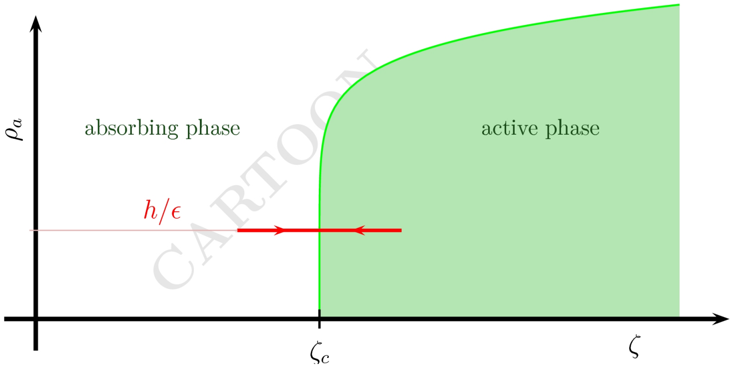
Dickman, Vespignani, Zapperi 1998

- SOC model: activity ρ_a leads to dissipation
- dissipation reduces particle density ζ
- density is reduced until system is inactive (ρ_a goes down)
  $\longrightarrow$ absorbing phase
- external drive increases particle density (ρ_a goes up)
  $\longrightarrow$ back to active phase

An SOC model can be seen as an AS model that drives itself into the inactive phase by dissipation ϵ and is pushed back into the active phase by external drive h.

$$\dot{\zeta} = h - \epsilon \rho_a \xrightarrow{\text{stationarity}} \rho_a = h/\epsilon$$

The Absorbing State Mechanism



Idea: SOC drives $h/\epsilon = \rho_a$ to 0 as $L \to \infty$

Leading orders: $h(L) = h_0 L^{-\omega}$ and $\epsilon(L) = \epsilon_0 L^{-\kappa}$

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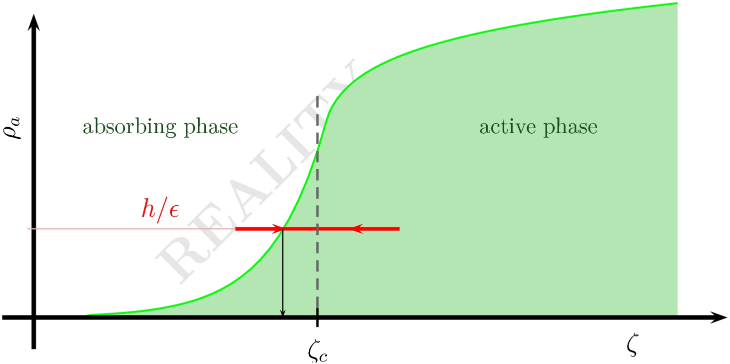
The Absorbing State Mechanism



Problem: SOC exponents would be affected by the way how driving and dissipation are implemented $\longrightarrow$ no universality.

Fey, Levine and Wilson suggest that critical point is not reached.

Outline

Field theory for SOC

The Manna Model

Field theoretic formulation of the time evolution of the Manna Model.
Note: Before taking any limits, this theory is *exact*.

- Continuum limit
- Simplify…
- Diagrams (meaning?, process?, tree level?)
- Renormalisation

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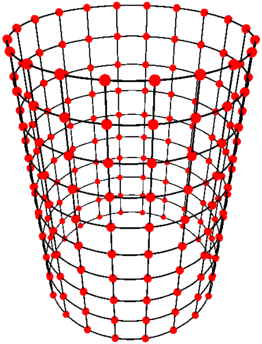
Simplification of the field theory

Bare propagators from field theory by inspection.

Simplification by considering periodic boundary conditions in $d-1$ directions. Surface appears in only one dimension.



Bare propagators

$$= \frac{1}{-\imath\omega + D(\mathbf{k}^2 + q_n^2)}$$

where $q_n = \frac{\pi}{L} n$ with $n = 1, 2, \ldots$

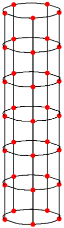
- $d-1$ dimensions can be treated the "usual" way.
- Usually, the gap in the propagator is the mass r_0 in

$$\frac{1}{-\imath\omega + D(\mathbf{k}^2 + r_0)}$$

  found by evaluating the inverse propagator at minimal momentum and frequency magnitude, $\mathbf{k} = 0$ and $\omega = 0$.
- Here, the gap is set by the minimum magnitude of q_n allowed. The effective mass is $q_1^2 = (\pi/L)^2$.

Bare propagators

Consider the system size as the effective mass of the system.
Expect convergence as circumference is increased; critical point controlled by height (L) only.



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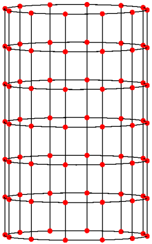

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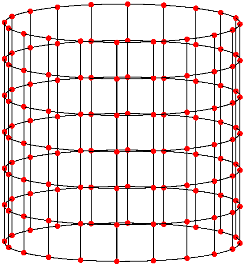

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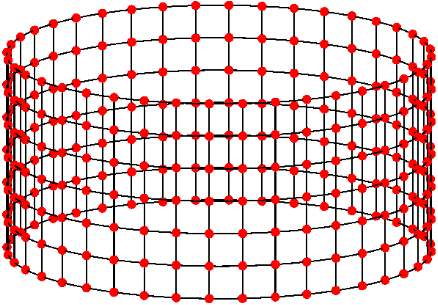

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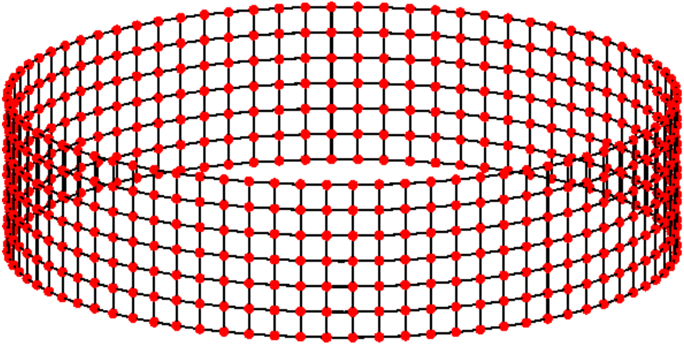

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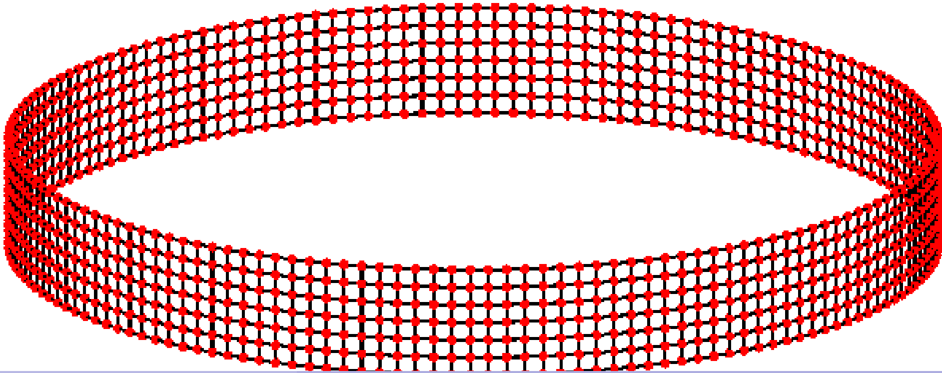


Bare propagators

Exact first moments

Circumference does not enter into first moment.
Avalanche size: Total activity (total number of charges).
In one dimension (continuum limit):

$$\langle s \rangle = \frac{1}{6} L^2$$

and $\langle s \rangle = \frac{1}{6}(L+1)(L+2)$ discretely. In higher dimensions:

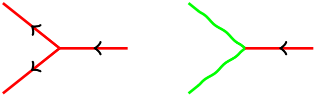
$$\langle s \rangle = \frac{d}{6} L^2$$

and $\langle s \rangle = \frac{d}{6}(L+1)(L+2)$ discretely.
Non-renormalisation of bare propagator!

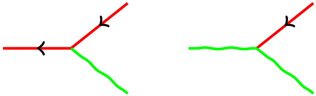
Vertices

The interaction vertices are

- Spontaneous branching and substrate deposition:



- Substrate interaction resulting in attenuation or deposition:



All relevant for $d \leqslant d_c = 4$. Loops occur.

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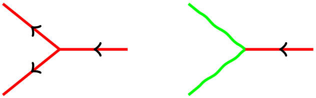
Vertices

The interaction vertices are

- Spontaneous branching and substrate deposition:



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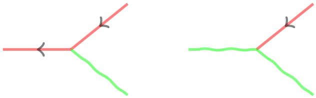
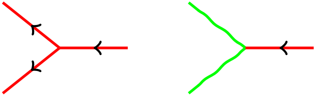


Only the former are relevant for $d > d_c = 4$; as in ϕ^4 the latter enter only for the lowest mode. No loops.

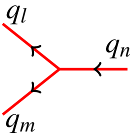
Tree level

Tree level becomes exact above $d_c = 4$. Two vertices are relevant there:



For example:

$$\langle s^2 \rangle = 2 \left(\frac{2}{L} \right)^3 \sum_{\substack{n,m,l \\ \text{odd}}} \frac{4}{q_l q_m}$$



$$\frac{2}{q_n} = \frac{d^3}{140} L^6$$

Higher order moments follow similarly.

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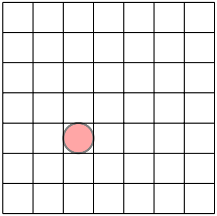
Tree level — applies above $d_c = 4$

Underlying process

Physics of the tree level diagrams (Manna Model above $d_c = 4$):

The **mean field theory of the Manna Model** is a **fair branching random walk on a lattice with open boundaries**.



In contrast to the usual *effective* mean-field theory of, the above identifies precisely which correlations and fluctuations are to be ignored.

Tree level — applies above $d_c = 4$

Underlying process

Physics of the tree level diagrams (Manna Model above $d_c = 4$):

Mean field theory of the Manna Model is a **fair branching random walk on a lattice with open boundaries**.

Avalanche moments can be calculated exactly.[1] Compare universal moment ratios to numerics at $d = 5$ (GP and Nguyen Huynh):

Observable	analytical	numerical (leading order)
$\langle s \rangle$	$(d/6)L^2 = 0.833\ldots L^2$	$0.83334(6)L^2$
$\left\langle s^3 \right\rangle \langle s \rangle / \left\langle s^2 \right\rangle^2$	$3.08754\ldots$	$3.061(5)$
$\left\langle s^4 \right\rangle \left\langle s^2 \right\rangle / \left\langle s^3 \right\rangle^2$	$1.6693\ldots$	$1.65(2)$
$\left\langle s^5 \right\rangle \left\langle s^3 \right\rangle / \left\langle s^4 \right\rangle^2$	$1.4005\ldots$	$1.38(3)$

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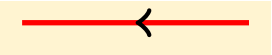
[1] Tedious! Use Mathematica!

The SOC mechanism — How does SOC work?

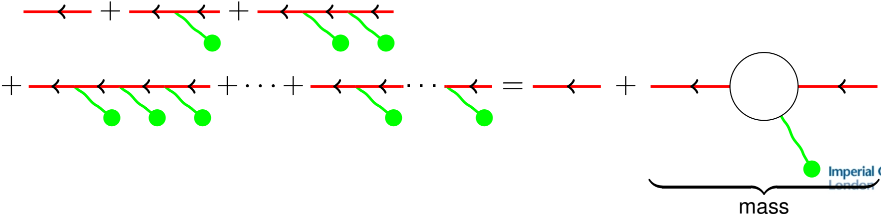
At criticality the renormalised mass vanishes:



$$= \frac{\delta_{nm}\delta(\omega'-\omega)\delta(\mathbf{k}'-\mathbf{k})}{-\imath\omega + D(\mathbf{k}^2+q_n^2)+r_0}$$

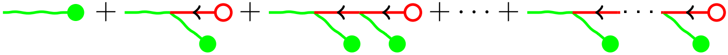
$\longrightarrow$ Why are the propagators massless?

Mass is attenuation (loss of activity). At tree level:



The SOC mechanism — How does SOC work?

Attenuation leads to deposition by the external drive — diagrams have that symmetry.
Density of particles in the substrate:



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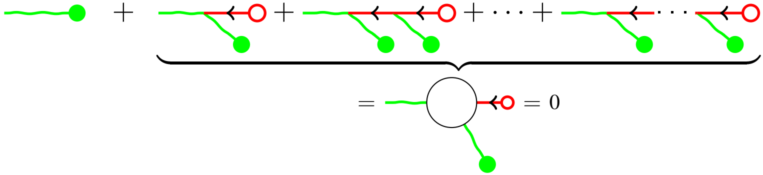
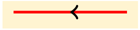


Additional deposition by external drive vanishes at stationarity.

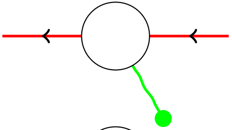
The SOC mechanism — How does SOC work?

At criticality the renormalised mass vanishes:

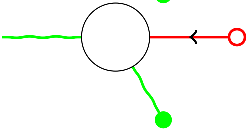


$$= \frac{\delta_{nm}\delta\!\!\!^{-}(\omega' - \omega)\delta\!\!\!^{-}(\mathbf{k}' - \mathbf{k})}{-\imath\omega + D(\mathbf{k}^2 + q_n^2) + r_0}$$

Propagator renormalisation including mass:



Additional deposition:



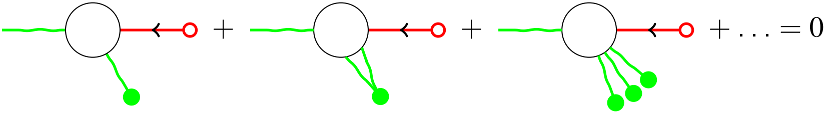
$= 0$

Only difference between the two diagrams: Left most vertex (coupling identical at renormalised and bare level).

The SOC mechanism

Beyond tree level

Argument extends beyond tree level and beyond one-point correlators of the substrate:



Propagator does not renormalise at any order.

This is why the bare propagator gives the exact average avalanche size as derived via random walker approach.

The SOC mechanism

So how does it work then?

Symmetry of vertices and stationarity.

- Mass is attenuation of activity.
- Conservation links attenuation to (additional) substrate deposition...
- or equivalently, symmetry of vertices equates mass terms of activity and substrate deposition terms.
- Additional substrate deposition vanishes *as we choose to consider stationarity.*
- Thus mass vanishes *in the particular ensemble*.
- **The activity propagator is not renormalised at any order.**

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What are the key findings?

- **Field theory for the Manna Model derived from microscopic rules**.
- Now we know **why** and **how** the propagator is massless.
- **Symmetry of vertices**, reflecting conservation (**conservation not necessary!**),
- …ensures that the renormalisation of the propagator vanishes at **stationarity**.
- Criticality is a matter of the (stationary) **ensemble**.
- Correlations in the bulk are non-trivial and shift the **local branching ratio**.
- Other mechanisms challenged: Absorbing states, sweeping across the critical point, Goldstone bosons, no criticality …

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Back to: Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Very likely so. Is a reliable test feasible?
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Probably.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks you!

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