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On the radial evolution of κ distributions of pickup protons in the supersonic solar wind

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Abstract It is well known that (pickup) ions in the inner heliosphere do not maintain Maxwellian distributions but tend to nonequilibrium distributions with extended suprathermal tails. Such states have been classified as quasi-equilibria which in many cases can well be described by so-called κ distributions. With the present study we start out from a phase space transport equation for pickup ions in the inner heliosphere that adequately describes the most important processes such as injection, convection, cooling, and diffusion in velocity space. Assuming that the underlying distribution functions are κ distributions, we proceed from this transport equation to a second-order moment, i.e., pressure equation which represents an ordinary differential equation for the κ parameter as function of heliocentric distance. This strategy allows one to describe the transition from an initial Vasyliunas-Siscoe distribution to κ distributions with gradually more pronounced suprathermal tails. While the velocity dependence of the velocity diffusion coefficient determines the systematic reduction of the parameter κ, the latter always has the (formal) asymptotic value $\kappa_\infty = 3/2$. This translates into values of $1.5 \leq \kappa_{TS} \leq 2.2$ in the upstream region of the (upwind) solar wind termination shock that defines the outer validity range of the model.

1. Introduction and Motivation

The transport of pickup ions (PUIs) has been a subject of investigations for more than about 40 years [see *Fahr*, 1973; *Holzer and Leer*, 1973; *Vasyliunas and Siscoe*, 1976] and has experienced a long chain of ever improving theoretical representations [see, e.g., *Isenberg*, 1987, 1995; *Rucinski et al.*, 1993; *Jokipii and Giacalone*, 1996; *Fichtner et al.*, 1996; *Schwadron et al.*, 1996; *Chalov and Fahr*, 1998; *Fahr*, 2007; *Fahr and Siewert*, 2008, 2010]. From the observational side, it has long been recognized that PUIs exhibit a core distribution below the injection velocity and an extended tail at higher velocities often characterized by a so-called (−5) velocity power law [*Gloeckler*, 2003; *Fisk et al.*, 2010; *Hill et al.*, 2009]. In consecutive papers by, e.g., *Fahr and Fichtner* [2011], *Fahr and Fichtner* [2012], and *Zhang and Schlickeiser* [2012], it has been attempted to describe this velocity space feature of PUIs as a phase-space transport phenomenon resulting from a joint action of injection, cooling, and velocity diffusion. It turned out that with appropriately shaped, but physically motivated velocity diffusion coefficients, one can convincingly reproduce the observed features of a core distribution and a tail distribution of PUIs.

Under similar conditions as for PUIs, namely, for transport with an injection, a velocity diffusion, and a relaxation process, it can be shown that the resulting typical nonequilibrium, quasi-stationary distribution functions are so-called κ distributions [*Hinton*, 1983; *Karney*, 1986; *Collier*, 1995; *Treumann*, 1999; *Treumann et al.*, 2004; *Shizgal*, 2007; *Livadiotis and McComas*, 2013]. Motivated by this and by the finding that κ distributions have been found to be reasonable approximations not only in the supersonic solar wind [e.g., *Scudder*, 1996; *Leubner*, 2004; *Pierrard and Lazar*, 2010] but also in the heliosheath [*Heerikhuisen et al.*, 2008; *Opher et al.*, 2013; *Zirnstein et al.*, 2014], we investigate—starting from a phase space transport equation like the one used by *Fahr and Fichtner* [2011] and specifying the velocity distribution as a κ function— whether one can derive a differential equation describing the evolution of the κ parameter with heliocentric distance.

A variation of the parameter κ with heliocentric distance has in fact been extracted from observational data but, to the best of our knowledge, only for electrons; see the papers by *Maksimovic et al.* [2005] and *Štverák et al.* [2009]. Also, these studies do not report the variation of the κ parameter beyond 4 AU.

Consequently, we present a first modeling attempt to explore how the quasi-stationary suprathermal PUI distribution function in the solar wind evolves with increasing heliocentric distance when assuming the κ model and whether that results in the κ values of about 1.6 to 2.0 typically used for simulations of the outer heliosphere [*Heerikhuisen et al.*, 2008; *Opher et al.*, 2013; *Zirnstein et al.*, 2014].

2. The Model

2.1. The Definition of κ Distributions

As pointed out by, e.g., *Treumann et al.* [2004], ions under the action of relaxation, pitch-angle diffusion as well as velocity diffusion processes tend to develop in velocity space to a quasi-equilibrium distribution function which is likely to be close to an isotropic κ distribution defined by [see, e.g., *Pierrard and Lazar*, 2010; *Livadiotis and McComas*, 2013]

$$f(r,v) = \frac{n(r)}{\pi^{3/2}} \frac{1}{\Theta^3 \kappa^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v^2}{\kappa\Theta^2}\right]^{-(\kappa+1)} \tag{1}$$

$$\equiv g_\kappa(r) \left[1 + \frac{v^2}{\kappa\Theta^2}\right]^{-(\kappa+1)} \tag{2}$$

with the (locally constant) parameter κ, n denoting the local particle number density depending on heliocentric distance r, and Θ describing a velocity spread of the core ions, i.e., the core width. $\Gamma(x)$ is the Gamma function.

2.2. The Phase Space Transport Equation for Protons

The transport of isotropically distributed suprathermal protons in the heliosphere is governed by three physical processes, namely, convection with the solar wind speed U, adiabatic or magnetic cooling [*Fahr*, 2007; *Fahr and Siewert*, 2008, 2010; *Fahr and Fichtner*, 2011]), and diffusion in velocity space. Assuming the so-called magnetic cooling, the equation describing the time evolution of the isotropic part of the proton phase space distribution $f(r,v,t)$ (in a mixed frame: solar rest frame in configuration and comoving frame in velocity space) reads as follows [*Isenberg*, 1987; *Chalov et al.*, 1995; *Fichtner et al.*, 1996; *Fahr and Fichtner*, 2011]:

$$\begin{aligned}
\frac{\partial f}{\partial t} = &\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 D_{vv}(r,v)\frac{\partial f}{\partial v}\right) && \text{(velocity diffusion)}\\
&+ \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^3 U}{r} f\right) && \text{(magnetic cooling)}\\
&- U\frac{\partial f}{\partial r} && \text{(convection)}\\
&+ S(r,v,t) && \text{(source)}
\end{aligned} \tag{3}$$

where radial symmetry has been assumed. The first term on the right-hand side describes diffusion in velocity space via the coefficient $D_{vv}(r,v)$. With the second and third terms, magnetic cooling and convection are treated for both of which we assume a constant solar wind speed U. The last term $S(r,v,t)$ represents the pickup proton injection function at heliocentric distance r, speed v, and time t. As we are not considering high-energy particles but limit the study to PUIs in the keV range, the equation is written in nonrelativistic notation.

As mentioned, we limit the study to the case of isotropic distribution functions, i.e., assuming that pitch-angle scattering is efficient like in, e.g., *Fahr et al.* [2012]. For anisotropic distributions the treatment would be more complex, see, e.g., *Isenberg* [1997] or *le Roux and Webb* [2009], but such is not required in the present context.

Let us now assume that the physical processes formulated in the phase space transport equation (3) locally lead to a quasi-equilibrium state like envisaged by *Treumann et al.* [2004], then this state should be sufficiently well described by a κ distribution according to equation (2) with $\kappa = \kappa(r)$. In the following, we determine the function $\kappa(r)$ that describes the joint operation of all relaxation, diffusion, and injection processes at heliocentric distance r. For that purpose we start from the kinetic transport equation (3) and calculate higher velocity moments. Proceeding along this line we derive an ordinary differential equation for the κ parameter as function of the heliocentric distance r.

3. Analytical Solutions for $\kappa(r)$

3.1. The Second-Order Moment Equation for the Pressure

Limiting the analysis to the steady state, i.e., $\partial f/\partial t = 0$, and assuming that the distribution of suprathermal pickup protons can be described at any location in the (upstream) heliosphere with a κ distribution, one obtains (with m denoting the proton mass) after the integration (using spherical coordinates in velocity space)

$$\frac{4\pi m}{3}\int_0^\infty [(\dots)\, v^2] v^2 \mathrm{d}v \tag{4}$$

from the transport equation (3)

$$0 = I_1(r) + I_2(r) - U\frac{\mathrm{d}P(r)}{\mathrm{d}r} + Q(r) \tag{5}$$

where we have defined on the right-hand side the following second-order velocity moments for the pressure $P(r)$, for the source $Q(r)$, as well as for the diffusion and cooling term $I_1(r)$ and $I_2(r)$, respectively:

$$P(r) = \frac{4\pi m}{3}\int_0^\infty f(r,v)\, v^4 \mathrm{d}v \tag{6}$$

$$Q(r) = \frac{4\pi m}{3}\int_0^\infty S(v,r)\, v^4 \mathrm{d}v \tag{7}$$

$$I_1(r) = \frac{4\pi m}{3}\int_0^\infty \frac{\partial}{\partial v}\left(v^2 D_{vv}\frac{\partial f}{\partial v}\right) v^2 \mathrm{d}v \tag{8}$$

$$I_2(r) = \frac{4\pi m}{3}\int_0^\infty \frac{\partial}{\partial v}\left(\frac{v^3 U}{r} f\right) v^2 \mathrm{d}v \tag{9}$$

These definitions imply a nonrelativistic treatment.

We consider a velocity diffusion coefficient with a separated r dependence and a power law v dependence in the following form:

$$D_{vv}(r,v) = D_0(r) v^\alpha \tag{10}$$

so that we obtain, after repeated partial integration, for $I_1(r)$

$$\begin{aligned}\frac{3}{4\pi m D_0} I_1(r) &= \underbrace{\left[v^{4+\alpha}\frac{\partial f}{\partial v}\right]_0^\infty}_{=0} - 2\int_0^\infty v^{3+\alpha}\frac{\partial f}{\partial v}\,\mathrm{d}v \\ &= -2\underbrace{\left[v^{3+\alpha} f\right]_0^\infty}_{=0} + 2(3+\alpha)\int_0^\infty v^{2+\alpha} f\,\mathrm{d}v \\ &= 2(3+\alpha)\int_0^\infty v^{2+\alpha} f\,\mathrm{d}v \end{aligned} \tag{11}$$

and for $I_2(r)$

$$\begin{aligned}\frac{3}{4\pi m} I_2(r) &= \underbrace{\left[\frac{v^5 U}{r} f\right]_0^\infty}_{=0} - 2\int_0^\infty \frac{v^4 U}{r} f\,\mathrm{d}v \\ \Rightarrow I_2(r) &= -\frac{2U}{r}\frac{4\pi m}{3}\int_0^\infty v^4 f\,\mathrm{d}v = -\frac{2U}{r}P(r)\end{aligned} \tag{12}$$

wherein the expressions in brackets vanish because of $v = 0$ at the lower limit and $f = 0$ as well as $\partial f/\partial v = 0$ at the formal, nonrelativistically valid upper limit $v = \infty$. These conditions at the upper limit correspond to the necessary transition of the distribution function from a power law to an exponential behavior at very high speeds.

Evidently, limiting the present study to positive power law exponents in equation (10), two cases can be distinguished, namely, $\alpha = 2$ and $0 < \alpha \neq 2$. Only for the latter case it is necessary to use an explicit form of the distribution function $f(r, v)$ in order to solve the integral $I_1(r)$. We consider both alternatives in the following subsections.

3.1.1. The Case $\alpha = 2$

With this choice we find $I_1(r) = 10D_0(r)P(r)$ and equation (5) reads

$$U\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \left(10D_0(r) - \frac{2U}{r}\right)P(r) + Q(r) \tag{13}$$

which represents a closed equation for the thermal pressure $P(r)$ if only the source function can be prescribed.

Note that for $\alpha = 0$, one obtains a similarly closed form because $I_1(r) = 2mD_0(r)n(r)$. This case, however, implies a velocity-independent diffusion coefficient D_{vv}, which can hardly be imagined to be a realistic choice.

3.1.2. The Case $0 < \alpha \neq 2$

In this case the situation is different because then $I_1(r)$ cannot be expressed naturally in terms of moments of the distribution function but requires an explicit expression for $f(r, v)$. Within the line of our approach we chose the κ distribution and obtain, using the auxiliary function $g_\kappa(r)$ defined in (2), and get from (11), see, e.g., *Gradshteyn and Ryzhik* [2007]:

$$\begin{aligned}\frac{3}{8(3+\alpha)\pi m D_0} I_1(r) &= g_\kappa(r)\int_0^\infty v^{2+\alpha}\left[1 + \frac{v^2}{\kappa\Theta^2}\right]^{-(\kappa+1)}\mathrm{d}v \\ &= g_\kappa(r)\frac{\Theta^2}{2}\frac{\Gamma(\kappa - \frac{1}{2} - \frac{\alpha}{2})\Gamma(\frac{3}{2} + \frac{\alpha}{2})}{\Gamma(\kappa)}(\kappa\Theta^2)^{(\frac{1}{2}+\frac{\alpha}{2})}\end{aligned} \tag{14}$$

The result is, in general, dependent on κ. For example, $\alpha = 1$ yields

$$I_1(r) = g_\kappa(r)\,\frac{16\pi m}{3}\,D_0(r)\frac{\kappa\Theta^4}{(\kappa - 1)} \tag{15}$$

The κ dependence is, however, weaker than seemingly implied by the latter expression because of the κ dependence of the function g_κ defined in (2) so that the limiting value (see, e.g., formula 6.1.47 in *Abramowitz and Stegun* [1964])

$$\lim_{\kappa\to\infty}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa - 1/2)}\,\frac{\kappa}{\kappa^{3/2}(\kappa - 1)} = 1 \tag{16}$$

is a reasonable approximation for $\kappa \gtrsim 3$, so that for $\alpha = 1$

$$I_1(r) \approx \frac{16\Theta m}{3\sqrt{\pi}}\,D_0(r)n(r) \tag{17}$$

holds. This fact motivates to investigate the asymptotic dependence of the general expression (14). Sorting $I_1(r)$ in κ-depending factors and others gives

$$I_1(r) = \frac{4(3+\alpha)m}{3\sqrt{\pi}}\,D_0(r)n(r)\Theta^\alpha h(\kappa) \tag{18}$$

with the auxiliary function

$$h(\kappa) = \kappa^{(\frac{\alpha}{2}-1)}\,\frac{\Gamma(\kappa+1)\Gamma(\kappa - \frac{1}{2} - \frac{\alpha}{2})\Gamma(\frac{3}{2} + \frac{\alpha}{2})}{\Gamma(\kappa - \frac{1}{2})\Gamma(\kappa)} \tag{19}$$

that has the asymptotics

$$\lim_{\kappa\to\infty} h(\kappa) = \Gamma((1+\alpha)/2) \tag{20}$$

This limit is approached within a factor of two for $\kappa \gtrsim 3$. Consequently, we can approximately express the integral $I_1(r)$ in the form

$$I_1(r) \approx \frac{4(3+\alpha)m}{3\sqrt{\pi}} D_0(r)n(r)\Theta^\alpha \Gamma((1+\alpha)/2) \equiv J(r) \tag{21}$$

With this finding, equation (5) takes the form

$$U \frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{2U}{r} P(r) + J(r) + Q(r) \tag{22}$$

Equations (13) and (22) can formally be combined into

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = a_1(r)P(r) + a_0(r) \tag{23}$$

with

$$\begin{aligned} a_0(r) &= \frac{Q(r)}{U} \quad ; \quad a_1(r) = \frac{10D_0(r)}{U} - \frac{2}{r} \quad \text{for} \ \ \alpha = 2 \\ a_0(r) &= \frac{J(r)+Q(r)}{U} \quad ; \quad a_1(r) = -\frac{2}{r} \quad \text{for} \ \ 0 < \alpha \neq 2 \end{aligned}$$

3.2. The Differential Equation for $\kappa(r)$

By evaluating the thermal pressure $P(r)$ for κ distributions (leading to the integral occurring in equation (14) for $\alpha = 2$), one obtains its dependence on κ

$$P(r) = \frac{1}{2} m\, n(r)\Theta^2 \frac{\kappa(r)}{\kappa(r) - \frac{3}{2}} \tag{24}$$

where κ now has to be considered as a function of heliocentric distance r. In the framework of the κ model, the quantity Θ can be understood as the velocity spread of the ions in the central core population. If we treat PUIs with this approach, one should feel inclined to identify this spread with that of the freshly injected PUIs, i.e., with the solar wind bulk velocity U (comoving shell distribution), which, in a good approximation, can be treated as independent on heliocentric distance, i.e., $\Theta = \text{const}$. Note, that with this we implicitly also assume that the tail is not significantly populated by particles from the thermal solar wind background.

With these assumptions we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{m}{2}\Theta^2 \frac{\kappa(r)}{\kappa(r)-\frac{3}{2}} \frac{\mathrm{d}n}{\mathrm{d}r} + \frac{m}{2}\Theta^2 n(r) \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\kappa(r)}{\kappa(r)-\frac{3}{2}}\right] \tag{25}$$

$$= \frac{m\Theta^2\kappa(r)}{2\kappa(r)-3}\frac{\mathrm{d}n}{\mathrm{d}r} - \frac{3m\Theta^2 n(r)}{(2\kappa(r)-3)^2}\frac{\mathrm{d}\kappa}{\mathrm{d}r} \tag{26}$$

Inserting this on the left-hand side and (24) on the right-hand side of equation (23) yields

$$\frac{\mathrm{d}\kappa}{\mathrm{d}r} = \frac{\kappa}{3}(2\kappa - 3)\left[\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}r} - a_1 - \frac{a_0}{m\Theta^2 n}\frac{2\kappa-3}{\kappa}\right] \tag{27}$$

which is the first main result of the paper. This ordinary differential equation provides the evolution of κ with heliocentric distance.

The right-hand side can be written explicitly for the two cases distinguished above. First, approximating the pickup proton number density by [see, e.g., *Fahr and Ruciński*, 1999]

$$n(r) \approx \sigma n_{p0} r_0^2 n_H (r - r_E) \tag{28}$$

with the charge exchange cross section σ, the solar wind proton number density n_{p0} at a reference distance r_0, the (constant) hydrogen number density n_H and $r_E = 1$ AU, it follows that

$$\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}r} \approx \frac{2r_E - r}{r(r-r_E)} \tag{29}$$

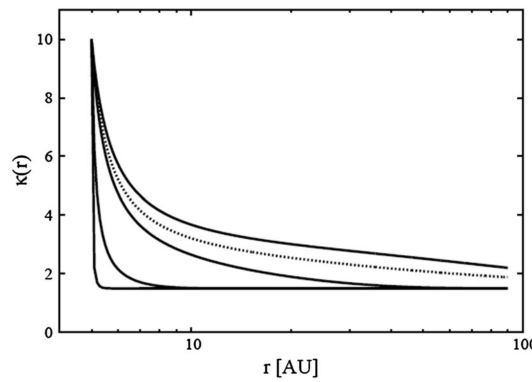

Figure 1. While the dotted line shows the solution (34) for $\alpha = 2$ and $D_0 = 0.01$, the solid lines are, in decreasing order, the solution (35) for $D_0 = 0.001, 0.01, 0.1$, and 1.0.

Second, setting [see, e.g., *Fahr and Ruciński*, 1999]

$$Q(r) \approx \frac{4\pi m}{3}\frac{\beta}{4\pi}U^2 = \frac{m}{3}\beta U^2 \tag{30}$$

with the ionization rate $\beta \approx \sigma U n_H n_{p0} r_0^2/r^2$ leads to

$$\frac{a_0}{m\Theta^2 n} \approx \begin{cases} \frac{4(3+\alpha)}{3\sqrt{\pi}} D_0(r)U^{\alpha-3}\Gamma\left(\frac{1+\alpha}{2}\right) + \frac{1}{r-r_E} & ; 0<\alpha\neq 2 \\ \frac{1}{r-r_E} & ; \alpha = 2 \end{cases} \tag{31}$$

so that, with these approximations, we finally obtain for the general case $0 < \alpha \neq 2$

$$\frac{\mathrm{d}\kappa}{\mathrm{d}r} = \frac{\kappa}{3}(2\kappa - 3)\left\{\frac{1}{r-r_E} - \frac{2\kappa-3}{\kappa}\left[\frac{1}{r-r_E} + \frac{4(3+\alpha)}{3\sqrt{\pi}} D_0(r)U^{\alpha-3}\Gamma\left(\frac{1+\alpha}{2}\right)\right]\right\} \tag{32}$$

and for the special case $\alpha = 2$

$$\frac{\mathrm{d}\kappa}{\mathrm{d}r} = \frac{\kappa}{3}(2\kappa - 3)\left\{\frac{1}{r-r_E} - \frac{10D_0}{U} - \frac{2\kappa-3}{\kappa(r-r_E)}\right\} \tag{33}$$

3.3. The Solution for $\kappa(r)$

Equations (32) and (33) have the solutions

$$\kappa(r) = \frac{3}{2}\frac{3\sqrt{\pi}(2r - r_E) + c_1(r)}{3\sqrt{\pi}r + c_1(r)} \quad ; \quad 0 < \alpha \neq 2 \tag{34}$$

and

$$\kappa(r) = \frac{3}{2}\frac{c_3(r)E(r) + 1}{c_3(r)E(r) + (1 + 5D_0[r - r_E]/U)} \quad ; \quad \alpha = 2 \tag{35}$$

with the auxiliary functions

$$c_1(r) = D_0 U^{\alpha-3}\Gamma\left(\frac{1+\alpha}{2}\right)4r(3+\alpha)(r - 2r_E) + \frac{c_2(r)}{2\kappa(r_i) - 3} \tag{36}$$

$$c_2(r) = 3\sqrt{\pi}(6r_i - 3r_E - 2\kappa(r_i)r_i + D_0 U^{\alpha-3}\Gamma\left(\frac{1+\alpha}{2}\right)4r_i(2\kappa(r_i) - 3)(3+\alpha)(2r_E - r_i) \tag{37}$$

$$c_3(r) = \frac{3 - 2\kappa(r_i) - 10(r_i - r_e)\kappa(r_i)D_0/U}{2\kappa(r_i) - 3} \tag{38}$$

$$E(r) = \exp(10[r - r_i]D_0/U) \tag{39}$$

In the above formula, $\kappa(r_i)$ denotes the value of κ at the inner boundary of integration at $r = r_i$. These analytical solutions for $\kappa(r)$ represent the second main result of the paper that is illustrated in the next section.

4. Illustrative Examples

The examples shown in Figures 1 and 2 illustrating the variation of $\kappa(r)$ were computed from equations (34) and (35) with an inner boundary $r_i = 5$ AU and $\kappa(r_i) = 10$. The solar wind speed U was normalized to unity (Figure 1) or set to $U = 2$ (Figure 2) and four different values of D_0 were chosen for each case.

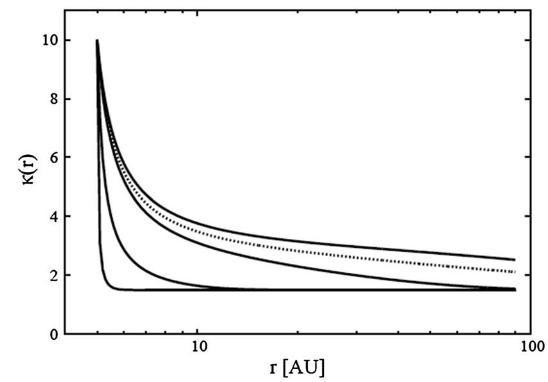

Figure 2. Same as Figure 1 but for $U = 2$.

Evidently, the solutions $\kappa(r)$ shown in Figures 1 and 2 reproduce the required behavior, namely, a reduction of comparatively high κ values to those between 1.5 and about 2. Also the decrease of the curves is, as expected, stronger for higher D_0, i.e., for stronger velocity diffusion.

5. Discussion and Conclusions

We have investigated the radial evolution of the velocity distributions of pickup protons in the solar wind by assuming that they are coconvected with the solar wind bulk under the simultaneous action of magnetic cooling, continuous injection at the solar wind speed, and turbulence-induced nonlinear velocity diffusion. We assumed further that their velocity distributions tend to be characterized by local quasi-equilibrium states that can well be represented by κ distributions.

Ascending by velocity space integration from the kinetic phase space transport equation to an associated second-order moment equation for the thermal pressure and explicitly using κ functions for the evaluation of the resulting velocity integrals we derived an ordinary differential equation for the κ parameter $\kappa = \kappa(r)$ as function of heliocentric distance r.

We have demonstrated that, depending on the magnitude and velocity dependence of the diffusion coefficient, one obtains reasonable radial variations of κ from comparatively high values in the inner heliosphere to values between 1.5 and about 2 at large distances. Interestingly, the asymptotic value $\kappa_\infty = 3/2$ corresponds to the (−5) velocity power law that is frequently observed for suprathermal ions [*Gloeckler*, 2003; *Fisk et al.*, 2010; *Hill et al.*, 2009] and that has commonly been used for simulations of the outer heliosphere.

In this paper we have especially focused on the treatment of PUIs that are locally injected into the bulk frame with the solar wind bulk velocity. By switching off the injection term and simply changing the core temperature, at present taken to be $\Theta^2 \simeq kT_p(r)/m$, where T_p denotes the core-Maxwellian temperature of the solar wind protons as given by, e.g., *Scime et al.* [1994], *Smith et al.* [2001], or *Fahr and Chashei* [2002], and k is the Boltzmann constant, we can also apply the identical treatment to normal solar wind ions.

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