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COMPRESSION ACCELERATION IN ASTROPHYSICAL PLASMAS AND THE PRODUCTION OF $f(v) \propto v^{-5}$ SPECTRA IN THE HELIOSPHERE

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ABSTRACT

We present a new derivation of the acceleration of fast charged particles by random compressions and expansions based on a quasilinear approximation applied to the Parker transport equation, and explore its consequences. This process has been suggested in a recent series of papers by Fisk & Gloeckler (F&G) as the origin of the quiet-time suprathermal ion population observed throughout the inner heliosphere with an omnidirectional distribution function close to the form $f(v) \propto v^{-5}$. Our derivation does not agree with a recent equation derived by F&G. We show that, while our equation conserves particles, the F&G equation does not. Solutions of the correct quasilinear equation are presented, which show that the compressive acceleration process does not produce power-law velocity spectra with indices less than (i.e., softer than) -3. We show that the transport equations for two other types of stochastic acceleration, by a spectrum of Alfvén waves and by transit-time damping of oblique magnetosonic waves, yield comparable acceleration rates but also do not produce power-law spectra with indices less than -3. Conversely, the process of diffusive shock acceleration, responsible for energetic storm particle events, corotating ion events and probably most large solar energetic particle (SEP) events, readily produces power-law velocity spectra with indices in a range including -5. It is suggested that the quiet-time suprathermal ion population is composed predominantly of remnant ions from these events as well as a contribution from impulsive SEP events.

Key words: acceleration of particles – interplanetary medium – turbulence

1. INTRODUCTION

In a recent series of publications and presentations, Fisk & Gloeckler have presented evidence for a ubiquitous suprathermal "tail" characterizing the ion distributions in the solar wind (Gloeckler et al. 1994; Fisk & Gloeckler 2006, 2007, 2008). In the frame of the solar wind the tail in velocity space is approximately isotropic, extends from the solar wind speed $v_{\rm sw}$ up to $\sim$30 times $v_{\rm sw}$ (in the approximate range 1 keV nucleon^{-1} –1 MeV nucleon^{-1}), and is generally characterized by an omnidirectional distribution function of the form $f(v) \propto v^{-5}$. These suprathermal ions occur in both fast and slow solar wind, and in the inner and outer heliosphere. Their composition is similar to that of the solar wind (Fisk & Gloeckler 2008). At the highest energies in this range of $\sim$1 MeV nucleon^{-1}, the observed distributions often exhibit an exponential rollover. In view of their ubiquitous occurrence in the heliosphere, often far removed from any observed shock waves, Fisk & Gloeckler (2006, 2007, 2008) have suggested that they are accelerated by stochastic acceleration due to turbulent compressions and rarefactions in the solar wind plasma.

However, this suprathermal tail also occurs in the vicinity of shock waves. The characteristic spectrum appears in the compression region bounded by the forward and reverse shocks in corotating interaction regions (CIRs) in the solar wind (Gloeckler et al. 1994), a region certainly containing ions accelerated by the process of diffusive shock acceleration (Fisk & Lee 1980). In addition, *Voyagers 1* and *2* have observed energetic ions downstream of the solar wind termination shock with a power-law spectral index close to -5 for energies up to a few MeV nucleon^{-1} (Decker et al. 2005, 2008; Stone et al. 2005, 2008). These "termination shock particles" (TSPs) have a spectral index reasonably close to that of the suprathermal ions observed by Ulysses within a heliocentric radius of 5 AU (e.g., Fisk & Gloeckler 2006). Fisk et al. (2006) have interpreted the TSPs as a suprathermal tail upstream of the termination shock, compressed adiabatically at the shock.

Mewaldt et al. (2007a) and Dayeh et al. (2009) have also investigated the energy spectra of suprathermal ions during quiet-times in order to eliminate as thoroughly as possible ions originating at the Sun or at interplanetary shock waves. Quiet-times are defined as periods with intensities below a prescribed low intensity for a prescribed period of time. Mewaldt et al. (2007a) based their study on *ACE* and *STEREO* data in the energy range 0.1–30 MeV nucleon^{-1}. Dayeh et al. (2009) based their study on *Wind* and *ACE* data from 1995 to 2007 in the energy range 0.04–2.56 MeV nucleon^{-1}. Dayeh et al. (2009) found that the energy power-law index of the differential intensity ranged from -1.27 to -2.29, a range that includes -1.5 corresponding to a -5 index for the dependence of the omnidirectional distribution function on particle velocity. Both studies found suprathermal tails during quiet times but with varying and often different spectral indices from the putative value -5 found by Fisk & Gloeckler. Dayeh et al. (2009) also found an ion composition distinct from the solar wind, which correlated with solar energetic particle (SEP) and energetic storm particle (ESP) events during solar maximum and with a mixture of solar wind and CIR-associated energetic ions during solar minimum.

Mewaldt et al. (2007b) performed another pertinent study based on *ACE* and GOES data, of the fluence energy spectra integrated over each year from 1997 to 2005. These spectra include all particles: quiet-time suprathermal particles, gradual SEPs presumably accelerated at shock waves, impulsive SEPs presumably accelerated at magnetic reconnection sites associated with a solar flare, ESPs accelerated at interplanetary shocks, particles accelerated at CIRs, anomalous cosmic rays presumably accelerated at the solar wind termination shock, and any "remnant" particles left over from the discrete events listed above. Interestingly, the observed energy spectral index of the fluence in the energy range from 0.1 to 2 MeV nucleon^{-1} for the eight different years varied between −1.3 and −2.1, which corresponds to a velocity index not far from the ubiquitous value −5 observed by Fisk & Gloeckler.

We conclude that observations do not always show a −5 spectrum, but that values near −5 are frequently observed. In view of this and the presence of these spectra both at shocks and during quiet times, we address the possible explanations of the suprathermal energy spectrum from a theoretical perspective, investigate the mechanism of stochastic acceleration in some depth, provide a critical assessment of the derivation provided by Fisk & Gloeckler (2008) for a statistical ensemble of compressions and rarefactions, and finally discuss the likely origin for these suprathermal ions.

2. POSSIBLE ACCELERATION MECHANISMS

Ultimately, because of the absence of collisions, the electric field of the plasma is responsible for all particle acceleration. In the solar wind, at the scales of interest here, the copious supply of electrons, which are extremely mobile and available to "short circuit" any significant electric field in the local plasma frame, ensures that in the coordinate frame where the plasma velocity is **V**, the electric field **E** may be expressed in terms of the magnetic field **B** by

$$\mathbf{E} \cong -c^{-1}\,\mathbf{V} \times \mathbf{B}. \tag{1}$$

There are locations in the space environment where this equation is violated: e.g., above the auroral oval in Earth's magnetosphere where electric fields are generated parallel to the magnetic field, at shock waves due to separation of electrons and ions in the shock ramp, or at high frequencies where electrostatic waves may be generated. However, in the bulk solar wind at scales relevant for the acceleration of energetic ions, Equation (1) is a valid approximation.

According to Equation (1) the electric field, and therefore significant particle acceleration, is intimately connected to spatial and/or temporal variations of **V**. If **V** is constant, the energy gain or loss is limited to that which occurs in the single transformation from the original frame to the plasma frame. However, if **V** is variable in space or time, energy gains may be larger. This intimate connection between acceleration and variable **V** underlies the origin of the two "classical" categories of acceleration described and developed by Fermi (1949, 1954): first-order Fermi acceleration and second-order Fermi acceleration. According to Equation (1), the scattering of particles in a parcel of plasma moving with velocity **V** is energy conserving when viewed in the frame of the plasma. On the one hand, first-order Fermi acceleration occurs when particles interact with parcels (or "clouds") that approach each other ($\nabla \cdot \mathbf{V} < 0$) so that the particles only experience "head-on" collisions with the individual parcels and gain energy as a result. On the other hand, second-order Fermi acceleration occurs when particles interact with both parcels that approach each other ($\nabla \cdot \mathbf{V} < 0$) and parcels that recede from each other ($\nabla \cdot \mathbf{V} > 0$); these interactions involve both "head-on" and "overtaking" collisions of the particles with the parcels, and therefore both energy gains and losses occur with a slight excess of gains. The presence of a mean magnetic field changes this slightly, as drift motions also play a role, but the result is basically the same. Clearly first-order Fermi acceleration is far more efficient than second-order Fermi acceleration; the former produces a systematic increase in energy, whereas the latter yields a particle "random-walk" in velocity space. At first thought it is difficult to imagine a configuration that results in sustained first-order Fermi acceleration. However, a shock wave provides precisely one such configuration and we shall return to shock waves in Section 6.

For particle speeds $v \gg |\mathbf{V}|\ (= V)$ and sufficient magnetic scattering to maintain a nearly isotropic ion distribution (appropriate for the suprathermal ion enhancements), an appropriate transport equation that addresses the acceleration of energetic particles due to plasma velocity variations is (Parker 1965)

$$\frac{\partial f}{\partial t} + (\mathbf{V} + \mathbf{V}_D) \cdot \nabla f - \nabla \cdot \mathbf{K} \cdot \nabla f - \frac{1}{3}\nabla \cdot \mathbf{V} v \frac{\partial f}{\partial v} = 0, \tag{2}$$

where $f(\mathbf{r}, t, v)$ is the omnidirectional distribution function, **K** is the spatial diffusion tensor, and $\mathbf{V}_D$ is the drift velocity (Jokipii et al. 1977). The acceleration arises from the term involving the v-derivative, which is proportional to $\nabla \cdot \mathbf{V}$. Equation (2) describes both first-order Fermi acceleration and drifts (at a shock, for example) and second-order Fermi acceleration (due to random fluctuations in **V** that produce regions of space with both $\nabla \cdot \mathbf{V} < 0$ and $\nabla \cdot \mathbf{V} > 0$). A subtlety arises in interpreting the acceleration due to $\nabla \cdot \mathbf{V} \neq 0$ in terms of the Fermi picture of "head-on" collisions or of "overtaking" collisions. With a magnetic field present, this term represents also that fraction of the energy gain or loss which is due to drift parallel to the motional electric field given by Equation (1), and that fraction is in general frame dependent (Jokipii 1979; Kota 1979). Thus, Equation (2), with $\mathbf{V}_D$ included, contains the general process of "diffusive shock acceleration" at both parallel (Krymsky 1977; Axford et al. 1978; Blandford & Ostriker 1978; Bell 1978) and quasi-perpendicular (Jokipii 1982, 1987) shocks.

The fluctuations of **V** need not only be due to different parcels or "clouds" of plasma with different velocities, as envisioned by Fermi, but may also be due to hydromagnetic waves propagating through the plasma, which satisfy Equation (1) but have wavelengths and phase speeds comparable with ion gyroradii and ion speeds, respectively. The interaction of these hydromagnetic waves with the ions involves resonances, which occur on the kinetic scale and are not described by Equation (2). The relevant transport equation in this case is based on the quasilinear equation describing the kinetic physics of the wave–particle interaction, to which we shall return in Section 5. In view of this and other important generalizations of Fermi's original ideas concerning particle acceleration due to a fluctuating velocity field $\mathbf{V}(\mathbf{r}, t)$, the terminology "second-order Fermi acceleration" is generally replaced by "stochastic acceleration."

3. STOCHASTIC ACCELERATION

Following Fisk & Gloeckler (2006, 2007, 2008; see also Ptuskin 1988; Jokipii et al. 2003; Webb et al. 2003), we consider the transport of ions described by Equation (2) in a plasma with small-amplitude velocity fluctuations $\delta\mathbf{V}(\mathbf{r}, t)$. Any average plasma velocity $\mathbf{V}_0$ is eliminated by expressing Equation (2) in the average plasma frame of reference. We then consider an ensemble of similar systems and require that the ensemble-averaged quantities describe a spatially homogeneous system, where averaged quantities are independent of position. Thus, $\langle f \rangle = f_0(v, t)$ and $\langle \mathbf{V} \rangle = 0$, where $\langle\rangle$ denotes ensemble average. Since the magnetic field is frozen into the plasma, the fluctuations in velocity $\delta\mathbf{V}(\mathbf{r}, t)$ will produce, in general, changes in the magnetic field and hence produce a fluctuating spatial diffusion tensor and guiding-center drift. We then set $\mathbf{K} = \mathbf{K}_0(v) + \delta\mathbf{K}(\mathbf{r}, t, v)$ and $\mathbf{V}_D = \delta\mathbf{V}_D(\mathbf{r}, t, v)$ with $\langle\delta\mathbf{K}(\mathbf{r}, t, v)\rangle = 0$ and $\langle\delta\mathbf{V}_D(\mathbf{r}, t, v)\rangle = 0$.

If we take the ensemble average of Equation (2), we obtain

$$\frac{\partial f_0}{\partial t} = \frac{1}{3v^2}\frac{\partial}{\partial v}\langle(\nabla\cdot\delta\mathbf{V})v^3\delta f\rangle, \tag{3}$$

where the fluctuations describe any particular realization of the ensemble. Subtracting Equation (3) from Equation (2) and retaining only those terms that are linear in the fluctuating quantities, we obtain the "quasilinear" approximation

$$\frac{\partial\delta f}{\partial t} - \nabla\cdot\mathbf{K}_0\cdot\nabla\delta f - \frac{1}{3}(\nabla\cdot\delta\mathbf{V})v\frac{\partial f_0}{\partial v} = 0. \tag{4}$$

We first consider the Green's function, $G(\mathbf{x}, t; \mathbf{x}', t')$, which satisfies

$$\frac{\partial G}{\partial t} - \nabla\cdot\mathbf{K}_0\cdot\nabla G = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{5}$$

For simplicity we take the diffusion tensor $\mathbf{K}_0$ to be isotropic. (Note, however, that $\mathbf{K}_0(v)$ is symmetric. Hence, if $\mathbf{K}_0(v)$ is *not* isotropic, it may be diagonalized by a coordinate rotation and made isotropic by rescaling the spatial coordinate in the direction i by $K_{0ii}^{-1/2}$.) The solution of Equation (5) is then

$$G(\mathbf{x}, t; \mathbf{x}', t') = [4\pi K(t-t')]^{-3/2}\exp\{-|\mathbf{x}-\mathbf{x}'|^2[4K(t-t')]^{-1}\} \tag{6}$$

for $t > t'$ and $G(\mathbf{x}, t; \mathbf{x}', t') = 0$ for $t < t'$, where K is the ensemble-average isotropic diffusion coefficient expressed without the zero subscript for convenience. Writing δf as the sum of the solution of the homogeneous equation with "initial" value $\delta f(\mathbf{x}, t_0) = \delta f_0(\mathbf{x})$ and the solution of the inhomogeneous equation based on Equation (6), we obtain the solution of Equation (4) as

$$\delta f(\mathbf{x}, t) = \int_{-\infty}^{\infty} d^3\mathbf{x}'\Bigg\{G(\mathbf{x}, t; \mathbf{x}', t_0)\delta f_0(\mathbf{x}') + \int_{t_0}^{t} dt' G(\mathbf{x}, t; \mathbf{x}', t')\frac{1}{3}(\nabla'\cdot\delta\mathbf{V}')v\frac{\partial f_0(v, t')}{\partial v}\Bigg\}, \tag{7}$$

where in $(\nabla'\cdot\delta\mathbf{V}')$ the gradient operates on the variable $\mathbf{x}'$. Substituting Equation (7) into Equation (3) we obtain

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\Bigg\{\frac{v^3}{3}\int_{-\infty}^{\infty} d^3\mathbf{x}' G\langle(\nabla\cdot\delta\mathbf{V})\delta f_0(\mathbf{x}')\rangle + \frac{v^4}{9}\int_{-\infty}^{\infty} d^3\mathbf{x}'\int_{t_0}^{t} dt' G\langle(\nabla\cdot\delta\mathbf{V})(\nabla'\cdot\delta\mathbf{V}')\rangle\frac{\partial f_0(v, t')}{\partial v}\Bigg\}, \tag{8}$$

where the arguments of the function G in the two terms have been suppressed. The first term in curly brackets involves the correlation of $\delta\mathbf{V}(\mathbf{x}, t)$ and $\delta f_0(\mathbf{x}')$, or equivalently the correlation of $\delta\mathbf{V}(\mathbf{x}, t)$ and $\delta\mathbf{V}(\mathbf{x}', t_0)$. If $t - t_0 \gg T$, where T is the correlation time, then this term is small and the influence of the initial condition is negligible. However, we also require that $|\delta\mathbf{V}|$ is sufficiently small that, during period T, $|\delta f| \ll f_0$ so the quasilinear approximation remains valid.[3] These are essentially the conditions for a Markov process. Under these conditions we may ignore the initial value term in curly brackets in Equation (8) and take $t_0 \to -\infty$. We may also replace the ensemble-averaged distribution $f_0(v, t')$ by $f_0(v, t)$, since the difference is third order in $|\delta\mathbf{V}|$. Under these conditions we obtain

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\Bigg\{\frac{v^4}{9}\int_{-\infty}^{\infty} d^3\mathbf{x}' \times \int_{-\infty}^{t} dt' G(\mathbf{x}, t; \mathbf{x}', t')\langle(\nabla\cdot\delta\mathbf{V})(\nabla'\cdot\delta\mathbf{V}')\rangle\frac{\partial f_0(v, t)}{\partial v}\Bigg\}. \tag{9}$$

Thus, the application of quasilinear theory to Equation (2) for this ensemble yields an equation describing spatially homogeneous particle transport in stochastic compressions and rarefactions, which is an isotropic diffusion equation in velocity space of the form

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 D\frac{\partial f_0}{\partial v}\right], \tag{10}$$

where D, the isotropic velocity–space diffusion coefficient, is given by

$$D = \frac{v^2}{9}\int_{-\infty}^{\infty} d^3\mathbf{x}'\int_{-\infty}^{t} dt' G(\mathbf{x}, t; \mathbf{x}', t')\big\langle(\nabla\cdot\delta\mathbf{V})(\nabla'\cdot\delta\mathbf{V}')\big\rangle. \tag{11}$$

D(v) is dependent on v through the explicit factor v^2 apparent in Equation (11) and the dependence of G on $K(v)$. Equation (10) has the standard form of a "continuity" equation expressing the transport of a conserved quantity, which in this case is the number density of particles. The particle flux is isotropic in velocity space and diffusive. This basic process of particle diffusion in velocity space due to random fluid compressions and rarefactions has been discussed previously by Ptuskin (1988), Jokipii et al. (2003), and Webb et al. (2003). The similar problem of particle acceleration at an ensemble of shocks separated by rarefactions was considered by Bykov & Toptygin (1981a, 1981b, 1993) and Schneider (1993).

For a simple illustrative example, take K to be independent of v, in which case D is proportional to v^2. The Green's function of Equation (10), obtained by adding $N(4\pi v_0^2)^{-1}\delta(v - v_0)\delta(t)$ to the right-hand side of Equation (10) and requiring $f_0(v, t < 0) = 0$, describes the time evolution of N particles per unit spatial volume distributed isotropically in velocity space with $v = v_0$ at $t = 0$. One might naively assume that the Green's function relaxes to the equilibrium power-law distribution given by requiring that the quantity in square brackets in Equation (10) is a constant, that is $f_0 \propto v^{-3}$ for the case that $D \propto v^2$. However, that assumption is incorrect because the resulting distribution has a divergent number density both as $v \to 0$ and as $v \to \infty$. The Green's function of Equation (10) with $D = D'v^2$ and D' constant is in fact given by

$$G(v, v_0, t) = \frac{N}{4\pi v_0^3}\frac{1}{(4\pi D't)^{1/2}}\left(\frac{v}{v_0}\right)^{-3/2}\exp\left(-\frac{9}{4}D't\right) \times\exp\left(-\frac{(\ln v/v_0)^2}{4D't}\right). \tag{12}$$

Equation (12) reveals a power-law distribution at long times with an index of $-3/2$, which "fills out" logarithmically at both small and large v as time increases, and which decreases exponentially

[3] It is important to note that neither the drift velocity nor the fluctuations in the diffusion tensor appear in the transport equation in this quasilinear limit. This is because neither affects the energy change directly. Both will, of course, enter in a more general formulation.

to accommodate the increasing extension of the distribution to small and large v.

An interesting more general case of Equation (10) occurs for $D(v) \propto v^{\alpha-2}$. Normalizing v to speed v_0, at which particles are injected at normalized time $\tau = 0$, the appropriate version of Equation (10) for this Green's function $G(v, \tau; 1, 0)$ is

$$\frac{\partial G}{\partial \tau} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^{\alpha}\frac{\partial G}{\partial v}\right] + \frac{1}{4\pi}\delta(v-1)\delta(\tau), \tag{13}$$

where G is normalized as

$$\int_0^{\infty} 4\pi v^2 G(v,\tau)dv = 1. \tag{14}$$

The solution of Equation (13) for integer and noninteger values of α is

$$G = \frac{v^{(1-\alpha)/2}}{|4-\alpha|4\pi\tau} I_{\pm p}\left[\frac{2v^{(4-\alpha)/2}}{(4-\alpha)^2\tau}\right]\exp\left[-\frac{1+v^{4-\alpha}}{(4-\alpha)^2\tau}\right], \tag{15}$$

where $p = (1-\alpha)/(4-\alpha)$ and $I_p(z)$ is the standard modified Bessel function of the first kind. The special cases for $\alpha = 3$ and $\alpha = 1$ are given by Tverskoi (1967) and Liu et al. (2007), respectively. It is clear that solution (12) for $\alpha = 4$ is a special case, which corresponds to both the index and the argument of the Bessel function in Equation (15) diverging. The behavior of G and the appropriate choice of the sign of the Bessel function index depend on the value of α. There are three distinct cases: (1) $\alpha > 4$; (2) $1 < \alpha < 4$; and (3) $\alpha < 1$. The appropriate choice of the Bessel function index in the three respective cases is (1) $(\alpha-1)/(\alpha-4) > 0$, (2) $(\alpha-1)/(4-\alpha) > 0$, and (3) $(\alpha-1)/(4-\alpha) < 0$. At small times G spreads as a Gaussian distribution in the variable $v^{(4-\alpha)/2}$ with $G \sim \exp[-(1-v^{(4-\alpha)/2})^2(4-\alpha)^{-2}\tau^{-1}]$. For large times case (1) behaves differently than the other two.

For case (1) the exponential rollover for $v < 1$ at small τ still occurs for large τ. However, the characteristic high-energy rollover speed $v_r \to \infty$ as $\tau \to (\alpha-4)^{-2}$. For larger times, $G \sim \tau^{-(2\alpha-5)/(\alpha-4)} v^{(1-\alpha)}$, multiplied by the exponential factor, which is a small correction at high speeds and provides the rollover at low speed. The power-law behavior yields a net (multiplied by $4\pi v^2$) positive flux of particles, which is independent of v for arbitrarily large v. This strange behavior results in an effective "free escape boundary" such that the net flux for any $\tau > 0$ does not vanish as $v \to \infty$. The exponential factor ensures a v-dependence of the net flux so that the left-hand side of Equation (13) is nonzero and in fact results in the power-law decrease with increasing τ noted above. The temporal decrease supplies the particles that extend the exponentially decaying "front" to smaller v and those that escape to large v. Thus, the solution exhibits a time-decaying approximate power law in v to arbitrarily large values, with an exponential rollover for small v. Note that since $\alpha > 4$ the power law in v results in a convergent number-density integral as $v \to \infty$. However, since that power-law decays with time, it is inconsistent with the positive escaping flux to $v \to \infty$. The correct interpretation of solution (15) in this case is that it is valid up to any finite v, which is then allowed to approach infinity. Since the power law in v would produce a divergence of the number density integral at small v, the power law has a low-energy rollover for any value of τ. Interestingly, the power-law slope changes discontinuously as α decreases to $\alpha = 4$, for which the slope at intermediate values of v is $-3/2$ as shown explicitly in Equation (12).

For cases (2) and (3) the exponential factor in Equation (15) gives a high-energy rollover at v_r, which increases with increasing τ. For larger τ the lead term of the series expansion of $I_p(z)$ yields $G \sim \tau^{-3/(4-\alpha)}$, multiplied by the exponential factor. The positive flux that arises from the exponential factor provides the particles that diffuse to higher v and causes the temporal decay. We note that G evolves to a flat distribution extending from $v = 0$ up to the rollover at $v \sim v_r$. The only difference between cases (2) and (3) is that the index of the Bessel function changes sign, which assures that G is nearly independent of v for larger τ and that the temporal decay rate decreases as α decreases below 1 and becomes negative. In the limit of very large negative α the advancing "front" of particles moves very slowly toward larger v where the diffusion coefficient is very small; the particles fill uniformly the region of large diffusion coefficient and only decay very slowly to fill the advancing front.

In summary, the Green's function solutions of Equation (13) given by Equation (15) evolve toward a power law for $\alpha > 4$ ($G \sim v^{1-\alpha}$) that extends to smaller v as time increases, whereas for $\alpha < 4$ they evolve to a uniform distribution that extends to larger v as time increases. The transition solution for $\alpha = 4$, given by Equation (12), has a power-law form at an intermediate range of v with a unique index $(-3/2)$, and has the unique property that the solution has advancing "fronts" at both large and small v. If the Green's functions are convoluted with an initial particle distribution $f_0(v_0, 0)$ to yield $f(v, t)$, then $f(v \gg v_0, t)$ exhibits the same behavior as $G(v \gg v_0, v_0, t)$.

We now evaluate D defined by Equation (11). For purposes of illustration, we may assume a simple form of the two-point correlation function in Equation (11), which is already assumed to be isotropic. We assume

$$\langle(\nabla\cdot\delta\mathbf{V})(\nabla'\cdot\delta\mathbf{V}')\rangle = \langle(\nabla\cdot\delta\mathbf{V})^2\rangle \times \exp[-|\mathbf{x}-\mathbf{x}'|^2\lambda^{-2} - (t-t')T^{-1}], \tag{16}$$

where λ is the correlation length and T is the correlation time. In reality, the function is probably different, but the form in Equation (16) is sufficient to illustrate our points. Substituting Equations (6) and (16) into Equation (11), we obtain

$$D = \frac{v^2}{9}\langle(\nabla\cdot\delta\mathbf{V})^2\rangle \int_0^{\infty} d\tau \exp\left(-\frac{\tau}{T}\right)\left(1+\frac{4K\tau}{\lambda^2}\right)^{-3/2}, \tag{17}$$

which may be evaluated as

$$D = \frac{v^2}{9}\langle(\nabla\cdot\delta\mathbf{V})^2\rangle\frac{\lambda^2}{2K}\left\{1-\pi^{1/2}\frac{\lambda}{(4KT)^{1/2}}\exp\left(\frac{\lambda^2}{4KT}\right) \times\left[1-\mathrm{erf}\left(\frac{\lambda}{(4KT)^{1/2}}\right)\right]\right\}. \tag{18}$$

Equation (17) or (18) yields for $T \ll \lambda^2/(4K)$

$$D = \frac{v^2}{9}\langle(\nabla\cdot\delta\mathbf{V})^2\rangle T \tag{19}$$

and for $T \gg \lambda^2/(4K)$

$$D = \frac{v^2}{9}\langle(\nabla\cdot\delta\mathbf{V})^2\rangle\frac{\lambda^2}{2K}. \tag{20}$$

Since $K(v)$ generally increases with increasing v, it is clear from Equation (17) that Dv^{-2} decreases with increasing v. Referring to Equations (11) and (13), $\alpha < 4$. Therefore, it would appear that the power-law solutions of Equation (13) at large times are not relevant.

However, another relevant case is that of continuous injection of particles with $v = v_0$ starting at $\tau = 0$. Although the spatial number density of particles in this case may increase without limit, it corresponds to a continuous injection of "seed" particles in the solar wind, for example, due to continuous ion pickup from the ionization of neutral atoms. The solution for $D(v) \propto v^{\alpha-2}$ is clearly given in this case by $H(v, \tau)$, where

$$H(v, \tau) = \int_0^\tau d\tau' G(v, \tau'; 1, 0) \tag{21}$$

and $G(v, \tau; 1, 0)$ is given by Equation (15). For $p > 0$, where p is the index of the Bessel function in Equation (15), the integral in Equation (21) converges as $\tau \to \infty$ to yield

$$\begin{aligned} H(\alpha > 1, v, \tau \to \infty) = {} & \frac{v^{(1-\alpha)/2}}{4\pi|4-\alpha|}[(1 + v^{4-\alpha}) - |1 - v^{4-\alpha}|]^p \\ & \times \frac{1}{p(2v^{(4-\alpha)/2})^p}. \end{aligned} \tag{22}$$

For both cases (1) and (2) Equation (22) yields

$$H(\alpha > 1, v > 1, \tau \to \infty) = (4\pi)^{-1}(\alpha - 1)^{-1} v^{1-\alpha}, \tag{23}$$

$$H(\alpha > 1, v < 1, \tau \to \infty) = (4\pi)^{-1}(\alpha - 1)^{-1}. \tag{24}$$

These solutions are finite because the amplitude adjusts itself so that the rate of injection equals the rate of escape in case (1), or that the rate of injection equals the rate at which particles fill out the divergent power law as $v \to \infty$ in case (2). For case (3) with $p < 0$ the integral in Equation (21) diverges as $\tau \to \infty$ at the upper limit of integration. For large τ the integral yields asymptotically

$$\begin{aligned} H(\alpha < 1, v, \tau \to \infty) \sim {} & \frac{(4-\alpha)^{2(1-\alpha)/(4-\alpha)}}{4\pi(1-\alpha)\Gamma(1-(1-\alpha)/(4-\alpha))} \\ & \times \tau^{(1-\alpha)/(4-\alpha)}. \end{aligned} \tag{25}$$

This distribution is independent of v below the rollover at $v \sim v_r$, where ${v_r}^{4-\alpha} \sim (4-\alpha)^2\tau$. The spatial number density N is given approximately by the phase-space density in Equation (25) multiplied by the volume in velocity space, $(4/3)\pi {v_r}^3$. Differentiating this quantity we obtain

$$\frac{dN}{d\tau} = -\frac{(4-\alpha)^3}{3(1-\alpha)^2\Gamma[(\alpha-1)/(4-\alpha)]}, \tag{26}$$

which equals unity, the particle injection rate, when α is large and negative so that the advancing "front" is steep. Thus, continuous injection of particles yields power-law spectra at large times if $\alpha > 1$. Accordingly, for a power-law dependence of $K(v)$ that increases with energy, the omnidirectional distribution function may have a power-law form with index in the range -3 to 0 for speeds $v < v_r$. For $\alpha < 1$ the index is 0 as described by Equation (25).

Altogether, for acceptable power-law velocity diffusion coefficients $D(v)$ and based on this analysis, power-law distribution functions at higher speeds are limited to indices "harder" than -3, and only for continuous injection. A power law with index -5 is simply not accessible for this mechanism.

In closing this section we should qualify its results. Although we have addressed primarily the specific problem addressed by Fisk & Gloeckler (e.g., particle transport based on the Parker equation, spatial homogeneity and isotropy of ensembled quantities, the limiting form of $D(v)$ for large $K(v)$ given by Equation (20)), we note that the actual problem could be more complicated. Adiabatic deceleration of the particles in the solar wind could modify the predicted distribution function. Also as v increases, $D(v)$ could transition from the form given in Equation (19) to that given in Equation (20) with the result that $D(v)$ does not exhibit a power-law dependence on v. A more subtle issue is that in the theory we have presented D does not vanish as K approaches zero as expected for an essentially adiabatic process. This behavior presumably arises from either the assumed form of the correlation function in Equation (16), the test particle approach in which the accelerating particles have no dynamical effects, or an inherent limitation of quasilinear theory. Finally, the diffusion Equation (10) exhibits a common feature of diffusion equations in that the distribution function, immediately after injection, has a finite value at arbitrarily large values of v. This fact is simply a limitation of the diffusion approximation and does not invalidate its utility. In our view these issues all remain open questions.

4. FISK–GLOECKLER DERIVATION OF STOCHASTIC ACCELERATION

Fisk & Gloeckler (2008, hereafter F&G) have presented a quite different derivation of the transport equation governing the stochastic acceleration of suprathermal ions due to random velocity fluctuations in the solar wind. This treatment builds on the more qualitative ideas of Fisk & Gloeckler (2006, 2007) and derives the transport equation resulting from these ideas. As in Section 3, their basic equation is Equation (2), the Parker transport equation. Since neither the derivation of Section 3 nor that of F&G includes the back reaction of the accelerating particles on the core plasma velocity fluctuations, these are both test-particle theories.

F&G initially work with the dependent variable E_v ($\propto v^4 f$), which is the differential energy density of the ion distribution function. Using this variable, Equation (2) becomes

$$\begin{aligned} \frac{\partial E_v}{\partial t} + \delta\mathbf{V} \cdot \nabla E_v - \nabla \cdot \mathbf{K} \cdot \nabla E_v + \frac{5}{3}(\nabla \cdot \delta\mathbf{V})E_v \\ - \frac{1}{3}(\nabla \cdot \delta\mathbf{V})\frac{\partial}{\partial v}(vE_v) = 0, \end{aligned} \tag{27}$$

which is identical with their Equation (3). Starting with our Equation (27), which includes all the temporal and spatial complexity of a "realization" of the velocity fluctuation field (initially an ensemble average is not employed), F&G make a number of ad hoc approximations: (1) the total energy flux $\mathbf{Q}_{\text{diff}}$ vanishes, where

$$\mathbf{Q}_{\text{diff}} = -\int_0^\infty dv \mathbf{K} \cdot \nabla E_v, \tag{28}$$

which appears to be inconsistent with the basic energization mechanism of a random compression combined with the irreversibility of diffusion away from the compressed parcel; (2) F&G replace the spatial diffusion term in our Equation (27) by a term representing relaxation toward an "equilibrium", which is not specified as a function of v or t; (3) F&G assume that the solution of Equation (27) is separable of the form

$E_v = g(v, t)h(\mathbf{r}, t)$, which again appears to be inconsistent with the basic process that the energization is due to localized regions of compression or rarefaction (that is, evolution in v is coupled to the specific location $\mathbf{r}$).

These approximations lead to an equation for $g(v, t)$, which eliminates the (5/3)-adiabatic-compression term and the convective term (presumably appropriate for a spatially homogeneous function $g(v, t)$), but it includes a remnant of the "relaxation" version of the spatial diffusion term. At this point F&G employ a quasilinear analysis based on an ensemble of similar systems, which is very similar in spirit to that presented here in Section 3. However, the spatial dependence of a realization of the ensemble is specified by the multiplicative function $h(\mathbf{r}, t)$, so that it is unclear how the ensemble average is defined. With a reasonable choice for the relaxation time, F&G obtain a temporal transport equation for g_0, the ensemble average of $g(v, t)$. With $g_0 = v^4 f_v$ that transport equation becomes

$$\frac{\partial f_v}{\partial t} = \frac{1}{v^4}\frac{\partial}{\partial v}\left[\frac{\langle \delta V^2\rangle}{9K} v \frac{\partial}{\partial v}(v^5 f_v)\right], \tag{29}$$

where δV is essentially the magnitude of the compressive velocity fluctuations and K is an effective isotropic spatial diffusion coefficient. Although f_v is defined in terms of the function $g(v, t)$, it should correspond to the ensemble average distribution function $f_0(v, t)$ defined in Section 3.

Equation (29) has similarities to Equation (9). The factors (1/9) and $\langle(\nabla \cdot \delta\mathbf{V})^2\rangle$ are common to both, although Equation (9) involves integrals dependent on correlation length and correlation time. However, the form of Equation (29) is fundamentally quite different from that of Equation (9). The fact that Equation (29) is not in the form of a continuity equation in velocity space means that it does not conserve particles. Particles will be created or destroyed at any arbitrary velocity. Also, the inverse dependence on $K(v)$ raises the question as to why the process is so efficient for small K. With the velocity correlation function given by Equation (16), Equation (11) yields an inverse dependence on $K(v)$ only when $T \gg \lambda^2/(4K)$, as shown by Equation (20). For $\lambda^2/(4K) \gg T$, the integral in Equation (11) is proportional to T and independent of K, as shown by Equation (19).

It may appear that a stationary solution of Equation (29) is $f_v \propto v^{-5}$. However, as discussed in Section 3, this solution is divergent as $v \to 0$ and is therefore not acceptable. A rollover at small v would be required, which forces an acceptable solution to depend on time.

More importantly Equation (29) is seriously flawed in that it does not conserve particles as mentioned above. The number density of the energetic particles is

$$N = \int_{v_0}^{\infty} 4\pi v^2 f_v(v), \tag{30}$$

where v_0 is a low speed threshold below which the Parker transport equation is not formally valid. The number density moment of Equation (29) yields after two integrations by parts

$$\frac{dN}{dt} = -\frac{4\pi\langle \delta V^2\rangle v_0^3}{9K(v_0)}\left[v_0\left(\frac{\partial f_v}{\partial v}\right)_{v=v_0} + 7f_v(v_0)\right] + \frac{2\langle \delta V^2\rangle}{9}\int_{v_0}^{\infty} dv\left[\frac{2}{K} + \frac{1}{K^2}v\frac{dK}{dv}\right]4\pi v^2 f_v(v). \tag{31}$$

Although it does not represent a diffusive flux, the first of the two terms on the right-hand side of Equation (31) presumably represents the change in the number density due to the transport of particles across $v = v_0$. However, the second term is in general not zero and represents a spontaneous production or loss of particles throughout velocity space. The only exception to the spontaneous production or loss occurs if $K \propto v^{-2}$, an unlikely velocity dependence of the spatial diffusion coefficient. The two terms in Equation (31) cannot in general cancel each other. This may be seen by considering the initial-value problem with particles initially localized in velocity space with speeds $v \gg v_0$. The second term immediately yields a growth or decay rate for the number density, while the first term is limited in magnitude by the number of particles that can diffuse to the low speed threshold in a short amount of time. This argument may be strengthened by considering the choice that $K \to \infty$ for speeds $v \sim v_0$ so that no particles can diffuse in velocity to v_0.

5. OTHER VARIATIONS OF STOCHASTIC ACCELERATION

Stochastic acceleration does not only arise from the interaction of particles with a random ensemble of large-scale compressions and rarefactions as described by Equation (9). It may also occur due to physical processes not included in Parker's Equation (2). One other form of stochastic acceleration occurs via the interaction of particles with the electric field fluctuations of an ensemble of smaller-scale waves propagating in various modes and directions. A common version of this process involves right-hand and left-hand circularly polarized hydromagnetic waves propagating parallel and antiparallel to the ambient magnetic field. Bogdan et al. (1991), for example, derive a transport equation in this case of the form presented in Equation (10) with D given by

$$D = \pi\left(\frac{qV_A}{mc}\right)^2\int_{-1}^{1} d\mu \frac{1-\mu^2}{v|\mu|}\frac{I_+(k_r)I_-(k_r)}{I_+(k_r)+I_-(k_r)}, \tag{32}$$

where q and m are the charge and mass of the energetic ion species, V_A is the Alfvén speed, $I_\pm(k)$ is the intensity of waves propagating parallel (+) or antiparallel (−) to the ambient magnetic field, μ is the cosine of the ion pitch angle, k_r is the low-wave-frequency version of the cyclotron-resonant wavenumber given by $k_r = \Omega/(v\mu)$, and Ω is the ion cyclotron frequency. Note that if all waves propagate in the same direction then $D = 0$. The requirement for stochastic acceleration to occur is that a particle with specified velocity may interact with waves of different phase speeds.

Another form of stochastic acceleration arises from the interaction of ions with the parallel magnetic field fluctuations of magnetosonic waves propagating at an oblique angle to the ambient magnetic field (Fisk 1976; Lee & Völk 1975). Associated with "transit-time" damping of the waves, the ions diffuse in their velocity component parallel to the ambient field through their Landau resonance with the waves. Qualitatively, the resonant ions are accelerated or decelerated toward the parallel wave phase speed ($\omega/k_\parallel$) by mirroring with constant first adiabatic invariant in the "magnetic bottles" associated with the compressive wave. Pitch-angle scattering in the associated Alfvénic turbulence ensures the near isotropy of the ion distribution function.

Lee & Völk (1975) give the general quasilinear velocity–space diffusion equation for the interaction of ions with magnetosonic waves. Retaining only the Landau resonance, which describes the "transit-time" acceleration process, and restricting the treatment to low frequency waves for which the Bessel function may be replaced by its small-argument value, we obtain

$$\frac{\partial F_0}{\partial t} = \frac{\partial}{\partial v_z}\left[\frac{\pi}{2}\int d^2\mathbf{k}_\perp \frac{k_z^2 k_\perp^2}{k^2}\frac{v_\perp^4}{|v_z|}\frac{I_M(k_z = \omega_M v_z^{-1}, \mathbf{k}_\perp)}{B_0^2}\frac{\partial F_0}{\partial v_z}\right], \tag{33}$$

where $F_0(\mathbf{v}, t)$ is the ensemble-averaged velocity–space distribution function, the z-direction and "$\perp$" refer to the ambient field, $\omega_M(\mathbf{k})$ is the wave frequency, and $I_M(\mathbf{k})$ is the magnetosonic wave intensity with $\int d^3\mathbf{k} I_M(\mathbf{k}) = \langle|\delta \mathbf{B}_M|^2\rangle$. Adding pitch-angle scattering in Alfvénic turbulence, which ensures near isotropy of the distribution, we obtain the equation satisfied by the omnidirectional distribution function as

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\frac{\pi}{4}\int_{-1}^{1} d\mu\mu^2 \int d^2\mathbf{k}_\perp \frac{k_z^2 k_\perp^2}{k^2}\frac{v_\perp^4}{|v_z|} \times \frac{I_M(k_z = \omega_M v_z^{-1}, \mathbf{k}_\perp)}{B_0^2}\frac{\partial f_0}{\partial v}\right]. \tag{34}$$

Equation (34) exhibits the standard form of isotropic diffusion in velocity space with the attendant conservation of particles.

The diffusion coefficient for v_z in Equation (33) satisfies approximately

$$D_{v_z v_z} \sim \frac{k_z^2 k_\perp^2 v_\perp^4}{k^2 \omega_M}\left(\frac{\delta B_M}{B_0}\right)^2 = \frac{k_z^2 v_\perp^4}{\omega_M}\left(\frac{\delta B_{M,z}}{B_0}\right)^2. \tag{35}$$

Equation (35) may be derived in a simpler fashion. Conservation of the first adiabatic invariant yields

$$\frac{dv_z}{dt} = -\frac{v_\perp^2}{2B}\frac{d\delta B_z}{dz} \tag{36}$$

or, approximately,

$$\Delta v_z \sim v_\perp^2 k_z (\delta B_z / B_0)\Delta t, \tag{37}$$

where we ignore numerical factors. The relevant diffusion coefficient is then given by

$$\left\langle \frac{\Delta v_z \Delta v_z}{\Delta t}\right\rangle \sim \frac{k_z^2 v_\perp^4}{\omega_M}\left(\frac{\delta B_z}{B_0}\right)^2, \tag{38}$$

where we set $\Delta t \sim {\omega_M}^{-1}$. Equation (38) is identical to Equation (35).

The functional form of $D(v)$ for Alfvén wave stochastic acceleration can be estimated from Equation (32). Spacecraft measurements in the solar wind show that $I(k) \propto k^{-\gamma}$, where typically $1 < \gamma < 2$ (Coleman 1968). Since $k_r \propto v^{-1}$ we expect $D \propto v^{\gamma-1}$, which implies in Equation (13) that $2 < \alpha < 3$. For transit-time acceleration of ions by magnetosonic waves we investigate the form of $D(v)$ in Equation (34), which includes efficient pitch-angle scattering by Alfvén waves. The Landau resonance requires that $k_z = \omega_M / v_z$ so that the integrand of D includes ${v_\perp}^4/|v_z|^3$. The v_z dependence of I_M is more difficult to determine. If refraction and/or turbulent processes are efficient at replenishing the highly oblique waves that are damped by the high-v ions, then I_M should be nearly isotropic in $\mathbf{k}$-space and insensitive to v_z. In this case $D \propto v$ and, with regard to Equation (13), $\alpha = 3$.

6. ORIGIN OF THE v^{-5} SPECTRUM

At this point, after discussing several acceleration mechanisms, it is worthwhile to speculate about the origin of the ions observed throughout the heliosphere with omnidirectional distribution approximately proportional to v^{-5}. Presumably the three stochastic acceleration mechanisms described above are independent processes. Ignoring small numerical factors and distinctions between the components of $\mathbf{k}$ and $\mathbf{v}$ parallel and perpendicular to the ambient magnetic field, the timescales of the three acceleration processes described by Equations (9) (including both limits described by Equations (19) and (20)), (32), and (34) or (35) are, respectively

$$t_{C1}^{-1} \cong k^2 \delta V_C^2 T/9, \tag{39}$$

$$t_{C2}^{-1} \cong k^2 \delta V_C^2 \lambda^2 K^{-1}/18, \tag{40}$$

$$t_A^{-1} \cong \pi\omega_A (V_A/v)(\delta B/B_0)^2, \tag{41}$$

$$t_T^{-1} \cong \omega_M (\delta B_z/B_0)^2, \tag{42}$$

where the subscripts C, A, and T refer to "compression," "Alfvén" and "transit-time," δV_C only includes the compressive part of δV, ω_A, and ω_M are characteristic Alfvén and magnetosonic frequencies, and δB in Equation (41) is restricted to the cyclotron resonant frequency range. We assume that K is dominated by its quasilinear component parallel to the magnetic field so that

$$K = v^2 \Omega^{-1}(B_0/\delta B)^2. \tag{43}$$

In addition, we take the compressive power to be reduced by a factor of 0.1 (Coleman 1968) so that $(\delta V_C)^2 = 0.1\ (\delta V)^2$ and $(\delta B_z)^2 = 0.1\ (\delta B)^2$. We also take $(\delta V/V_A)^2 = (\delta B/B_0)^2$. Then Equations (39)–(42) may be rewritten as

$$t_{C1}^{-1} \cong 0.1\Omega(\omega_A/\Omega)(\delta B/B_0)^2, \tag{44}$$

$$t_{C2}^{-1} \cong 0.1\Omega(V_A/v)^2(\delta B/B_0)^4, \tag{45}$$

$$t_A^{-1} \cong \pi\Omega(V_A/v)^2(\delta B_{\mathrm{res}}/B_0)^2, \tag{46}$$

$$t_T^{-1} \cong 0.1\Omega(\omega_A/\Omega)(\delta B/B_0)^2, \tag{47}$$

where we have also taken $\omega_M = \omega_A$, $k\lambda = 2\pi$, and $\omega_A T = 2\pi$. Admittedly in this brief discussion we have ignored different dominant frequencies for the different processes, and the extent of the cyclotron resonant frequency range in Equation (46). Nevertheless, these acceleration rates are comparable. Interestingly the compressional acceleration with velocity diffusion coefficient proportional to K^{-1} described by Equation (45), which is the primary subject of this paper, appears likely to have the slowest rate of acceleration. It is also worth noting that, for the scalings we have chosen, $T \gg \lambda^2/(4K)$ for the suprathermal ions we are considering so that Equation (45), and not Equation (44), should be used to describe stochastic acceleration due to random compressions and rarefactions. It should also be noted that application of these formulae depends on the values of the parameters at the site considered.

An important issue for these stochastic acceleration mechanisms is that they are limited by the power in the fluctuations or by how rapidly that power can be replenished. Generally the fluctuation power is limited by the power in the ambient magnetic field, which is, for example, small in the outer heliosphere. However, the major weakness of stochastic acceleration as a source of the quiet-time suprathermal ions is that for $\alpha < 4$ the distribution for impulsive particle injection relaxes to a value independent of v for $v < v_r$ that decreases with time as v_r increases with time. For $\alpha < 4$ and continuous particle injection the distribution relaxes to a power law with index greater ("harder") than -3. As we have shown in Sections 3 and 5, for all three types of stochastic acceleration $\alpha < 4$.

The other possible acceleration mechanism for these ions is shock acceleration. An advantage of shock acceleration is that, unlike stochastic acceleration, it naturally produces power-law distributions. A stationary planar shock accelerates low energy ions to an omnidirectional distribution function with a power law in v with index $-3X/(X-1)$, where X is the shock compression ratio. An index of -5 requires $X = 2.5$, which is a reasonable compression ratio for the large shocks in interplanetary space that produce most of the energetic particles, but which has not yet been shown to be a particularly "favored" value. Actually the ions respond to the average wave-frame compression ratio. Since the wave-frame compression ratio is thought to be less than the plasma compression ratio, an observed index of -5 would arise for a shock with a compression ratio somewhat larger than 2.5. Another advantage of shocks may be that more energy is available to convert to the energetic particles.

In our opinion the most promising explanation for the origin of the ions is a superposition of remnant ions from previous gradual SEP, impulsive SEP, and corotating ion events, with possible re-acceleration of ions by solar wind turbulence as described in this paper. Of course adiabatic deceleration in the expanding solar wind systematically reduces ion energy and partially counteracts the acceleration mechanisms. The *ACE* fluence spectra of ions summed over all these particle populations have spectral indices rather close to -5 (Mewaldt et al. 2007b). In addition, the composition of the quiet-time suprathermal particles investigated by Desai et al. (2006) and Dayeh et al. (2009), featuring SEP and ESP abundances during solar maximum, and CIR and solar wind abundances during solar minimum, supports this explanation.

The so-called TSPs observed downstream of the termination shock by both *Voyager 1* and *Voyager 2* deserve additional discussion. They also exhibit a speed power-law index close to -5, particularly those observed by *Voyager 1*. Fisk et al. (2006) interpret this particle distribution as a result of adiabatic compression of the ubiquitous heliospheric suprathermal particles at the termination shock, which preserves their spectral index of -5. Apart from the issue of why these ions do not participate in the required increase in entropy at the shock and the established process of diffusive shock acceleration, the uniform upstream intensity of advected suprathermal ions (on which are superposed the TSPs escaping from the downstream TSPs to create the termination shock foreshock) is far too small to support the hypothesis of Fisk et al. (2006). This issue has recently been studied by Giacalone & Decker (2010), who come to a similar conclusion. A more reasonable explanation is that the TSPs are predominantly the higher energy interstellar pickup ions in the solar wind, accelerated locally at the termination shock with a downstream power-law index in speed determined by the shock compression ratio. The observed index is consistent with the observed shock compression ratio in the range 2–3. In contrast with the smoothly varying foreshock intensity predicted by the planar stationary theory of diffusive shock acceleration, the upstream TSPs exhibit extreme irregularity that could possibly have been anticipated based on the irregularities at quasi-perpendicular shocks observed at 1 AU (van Nes et al. 1984). The irregularities arise from the dominant contribution of magnetic field line wandering to the spatial diffusion coefficient parallel to the shock normal so that the ion injection rate and the upstream particle intensity are very sensitive to the concurrent magnetic field configuration and its spatial and temporal variations.

7. CONCLUSIONS

We have addressed several aspects of how to accelerate the ubiquitous suprathermal ion populations observed throughout much of the heliosphere at quiet times, whose omnidirectional distribution function in the energy range from $\sim$1 keV nucleon^{-1} to $\sim$1 MeV nucleon^{-1} exhibits a power-law dependence on v with an index which is often close to -5. We first showed how stochastic acceleration by random compressions and rarefactions follows from the Parker transport equation by applying quasilinear theory to an ensemble of velocity fluctuations in the solar wind. In the simplest case of an isotropic, spatially homogeneous ensemble the resulting transport equation is an isotropic diffusion equation in velocity space, a hallmark of stochastic acceleration (Davis 1956; Parker & Tidman 1958). The velocity transport equation derived by Fisk & Gloeckler (2008), also based on the Parker transport equation and a number of ad hoc assumptions, is not of this form. We have identified and critiqued the ad hoc assumptions and shown that the F&G velocity transport equation does not conserve particles and is therefore unacceptable.

The origin of the frequently observed power law with index -5 has been ascribed by Fisk & Gloeckler (2006, 2007, 2008) to compressional stochastic acceleration of solar wind ions. Gloeckler et al. (1994) has also noted that a very similar spectrum of energetic particles occurs between the two shocks bounding CIRs, and Fisk et al. (2006) argue that the similar spectrum of energetic particles downstream of the solar wind termination shock is due to adiabatic compression at the shock of the suprathermal ion population in the outer heliosphere.

It is often assumed that stochastic acceleration results in power-law energy spectra at long times (e.g., F&G). This assertion may stem from the work of Fermi (1949) on cosmic ray acceleration by "second-order Fermi acceleration" in which a power law was derived by balancing the average rate of particle energy gain with the rate of spallation reaction loss in the Galaxy. Or it may stem from the specific assumption that the long-time solution of the velocity transport equation including a loss rate is stationary and equidimensional, a consequence of which is a power-law spectrum. In contrast we have presented solutions to the family of isotropic diffusion equations with diffusion coefficients proportional to $v^{\alpha-2}$ for any real α, for both impulsive and continuous particle injection, and for an injected speed distribution proportional to $\delta(v - v_0)$. We have then shown for reasonable spatial diffusion coefficients $K(v)$ that $\alpha < 4$ and that the corresponding solutions evolve to a form of the omnidirectional distribution function approximately independent of v up to a rollover speed v_r, which increases with time, for impulsive injection, or a power law with index greater ("harder") than -3 for continuous injection. These solutions cannot produce a power-law distribution with an index close to -5. Interestingly, the solutions with $\alpha > 4$ do produce a power-law spectrum with index $1 - \alpha$ at long times (the index -5 would correspond to $\alpha = 6$), characterized by particle free escape to $v \to \infty$, but these solutions are not consistent with the form of the velocity diffusion coefficient.

We have also explored two other types of stochastic acceleration that in principle could be responsible for the acceleration of the suprathermal ions: stochastic acceleration by a spectrum of Alfvén waves propagating in different directions and "transit-time damping" of a spectrum of obliquely propagating magnetosonic waves. By comparing the magnitudes of the velocity diffusion coefficients $D(v)$ for these types with that for compressional stochastic acceleration, we find that they are all comparable. However, we also find that $\alpha < 4$ for these two types of stochastic acceleration so that also they do not produce power laws with indices "softer" than -3.

A further difficulty for stochastic acceleration is that the energy density in the velocity and magnetic field fluctuations must be larger than that of the energetic particles, or it must be readily replenished, for example, by turbulent cascade processes. This back reaction of the accelerating particles on the fluctuations has not been considered by us or F&G and is generally neglected in treatments of stochastic acceleration. An exception is the work of Bogdan et al. (1991) on stochastic acceleration of ions by a spectrum of Alfvén waves that are damped by the accelerating particles.

The process of diffusive shock acceleration readily produces power-law spectra with power-law indices insensitive to particle transport parameters. However, there is nothing special about the index -5, which obtains for the omnidirectional distribution function produced downstream of a stationary planar shock with a compression ratio of 2.5. Nevertheless, most shock-accelerated particles are produced by a few large ESP, and presumably SEP, events, and a compression ratio of 2.5 (or somewhat higher to account for the lower wave-frame compression ratio) is probably characteristic of the larger interplanetary shock events. (However, it must be noted that a fluence spectrum differs from the source spectrum by the energy dependence of the spatial diffusion coefficient.) Indeed the eight years of one-year-averaged *ACE* fluence spectra obtained by summing over all energetic particle populations each year (Mewaldt et al. 2007b) reveal differential-intensity energy spectral indices between -1.3 and -2.1 in the energy range 0.1–2 MeV nucleon^{-1}, not far from the value -1.5 corresponding to -5. Large ESP and SEP events dominate this energy range. These fluence spectra suggest that the quiet-time particle spectra are the remnants of those particles that dominate the *ACE* fluence spectra. Dayeh et al. (2009) have found that the quiet-time ion energy spectra in the energy range 0.04–2.56 MeV nucleon^{-1} exhibit spectral indices in the range from -1.27 to -2.29, which encompasses the much narrower range quoted by F&G and is nearly identical to the range of the *ACE* fluence spectra. In addition, Dayeh et al. (2009) have found that the composition of the quiet-time suprathermal ions corresponds to that of ESP/SEP composition during solar maximum and that of CIR-associated particles and the solar wind during solar minimum. This solar-cycle-dependent composition also supports the hypothesis that the quiet-time spectra are dominated by remnant suprathermal particles from previous ESP, SEP, and CIR events with at most a small contribution directly from the solar wind.

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