Particle pusher in ultra-strong electromagnetic fields

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Outline

1. Objectives & Methods
2. An ultra-relativistic pusher for the Lorentz force
3. Acceleration in a spherical wave
4. Application to a rotating dipole
5. Conclusions

1 Objectives & Methods

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Objectives & Methods

Physical challenges

- compute particle acceleration and radiation reaction in a realistic environment.
- evaluate the impact of radiation reaction on particle acceleration efficiency.
- follow accurately particle trajectories.

Methods

- design a particle pusher for ultra-strong fields based on analytical solutions of the reduced Landau-Lifshitz approximation (LLR, i.e. $\vec{E}$ and $\vec{B}$ constant). (Heintzmann & Schrüfer, 1973; Boghosian, 1987; Li et al., 2021)
- long term task : a fully electromagnetic Particle-In-Cell (PIC) code for ultra-strong fields and ultra-relativistic particles.

Exact solutions of Landau-Lifshitz equation

- The Landau-Lifshitz equation with 4-velocity u^i, electromagnetic tensor F^{ik}, particle charge and mass q, m, proper time τ

$$\frac{du^i}{d\tau} = \frac{q}{m} F^{ik} u_k + \frac{q\,\tau_m}{m} \partial_\ell F^{ik} u_k u^\ell + \frac{q^2\,\tau_m}{m^2} \left[F^{ik} F_{k\ell} u^\ell + (F^{\ell m} u_m)(F_{\ell k} u^k) \frac{u^i}{c^2} \right].$$

with the radiation damping time scale (for electrons)

$$\tau_m = \frac{q^2}{6\,\pi\,\varepsilon_0\,m\,c^3} = \frac{2}{3}\,\frac{r_e}{c} \approx 6{,}26 \cdot 10^{-24}\ \mathrm{s}.$$

Two important parameters of the problem
  - strength parameter
    $$a = \frac{\omega_{\mathrm{B}}}{\Omega} \gg 1$$
  - radiation damping parameter
    $$b = \Omega\,\tau_m \ll 1$$
- Exact solutions for u^i are known
  - in plane electromagnetic waves. (Piazza, 2008; Hadad et al., 2010)
  - in constant $\vec{E}$ and $\vec{B}$. (Heintzmann & Schrüfer, 1973; Boghosian, 1987; Li et al., 2021)

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The central idea for the Lorentz pusher

- analytical solution are simple in the frame where $\vec{E}$ and $\vec{B}$ are parallel

$$\frac{du^i}{d\tau} = \frac{q}{m} F^{ik} u_k$$

- in the Cartesian coordinate system, the electromagnetic field tensor is anti-diagonal and given by (along e_z)

$$F^{ik} = \begin{pmatrix} 0 & 0 & 0 & -E/c \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ E/c & 0 & 0 & 0 \end{pmatrix} .$$

- the 4-velocity evolution given in terms of the proper time τ according to

$$u^0(\tau) = \gamma_0 \, c \, \left[\mathrm{ch}(\omega_E \, \tau) + \beta_0^z \, \mathrm{sh}(\omega_E \, \tau)\right]$$
$$u^3(\tau) = \gamma_0 \, c \, \left[\mathrm{sh}(\omega_E \, \tau) + \beta_0^z \, \mathrm{ch}(\omega_E \, \tau)\right]$$
$$u^1(\tau) = \gamma_0 \, c \, \left[\beta_0^x \, \cos(\omega_B \, \tau) + \beta_0^y \, \sin(\omega_B \, \tau)\right]$$
$$u^2(\tau) = \gamma_0 \, c \, \left[-\beta_0^x \, \sin(\omega_B \, \tau) + \beta_0^y \, \cos(\omega_B \, \tau)\right].$$

- integrate with respect to the proper time to find the trajectory of the particle.

The algorithm for the Lorentz pusher

- the idea is to switch to the frame where $\vec{E}$ and $\vec{B}$ are parallel by a Lorentz boost (primed frame).

$$\mathbf{V} = \frac{\mathbf{E} \wedge \mathbf{B}}{E_0^2/c^2 + B^2} \quad ; \quad E^2 - c^2\,B^2 = E_0^2 - c^2\,B_0^2 \quad ; \quad \mathbf{E} \cdot \mathbf{B} = E_0\,B_0.$$

- this is always possible if the field is not null-like

$\Rightarrow$ special case to be treated separately.

- the field must be constant within one time step Δt.
- the analytical solution is simple in the new frame where $\vec{E}' \parallel \vec{B}'$, assumed to be along z'.
- it requires several Lorentz boosts and spatial rotations to bring the common axis along z'.
- straightforward extension to include radiation reaction by adding a prefactor to the 4-velocity components for the magnetic part and the electric part.
- need to solve a non-linear equation for imposing $\Delta\tau$ from Δt.

(Pétri, 2020)

Some time-dependent tests of the ultra-relativistic Lorentz pusher

A charge $-q$ orbiting around a fixed charge q.

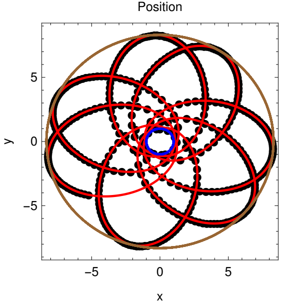

Fig. – Relativistic Kepler problem.

Periodic Lorentz factor variation with phase $\xi = k^i\, x_i = \omega\, t - k\, x$

$$\gamma(t) = 1 + a^2\,(1 - \cos\xi)$$

$$\gamma_{\rm max} = 1 + 2\,a^2$$

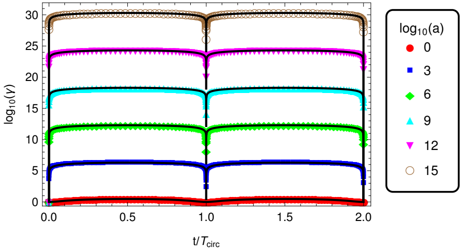

Fig. – Lorentz factor for circularly polarized plane wave.

- Full details and more tests in (Pétri, 2020).
- See (Tomczak & Pétri, 2020) for applications to neutron stars.
- The pusher works remarkably for test particles but what about full plasma simulations ?

$\Rightarrow$ design of a 1D fully relativistic electromagnetic PIC code as a proof of concept.

1D electromagnetic PIC implementation

Two counter-streaming relativistic beams with Lorentz factor each of $\Gamma_b = 10$.
Linear growth rate well known analytically

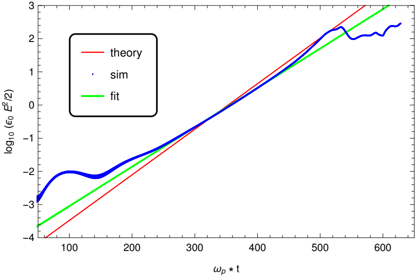

Fig. – Relativistic two-stream instability.

A simple 1D shock problem with bulk Lorentz factor Γ and strong magnetization

$$\sigma = \frac{B^2}{\mu_0 \, \Gamma \, n \, m \, c^2} .$$

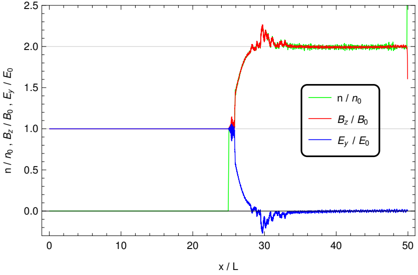

Fig. – Ultra-relativistic strongly magnetized shock with $\Gamma \approx 70$ and $\sigma \approx 70$.

- not yet satisfactory because we need to include radiation reaction damping.

⇒ use the reduced Landau-Lifshitz prescription à la (Heintzmann & Schrüfer, 1973).

Straightforward radiation reaction implementation

- Exact solution are easily construct in the frame where **E** and **B** are parallel.
- simply add a factor for the 4-velocity component associated to **E** and **B**.
- Lorentz force pusher straightforwardly modified.
- the exact solutions are for non null-like fields (Laue & Thielheim, 1986)

$$u^0(\tau) = \frac{1}{\sqrt{A + B\, e^{-2\,\tau_0\,(\lambda_E^2+\lambda_B^2)\,\tau}}}\,\gamma_0\, c\,\left[\mathrm{ch}(\omega_E\,\tau) + \beta_0^z\,\mathrm{sh}(\omega_E\,\tau)\right]$$

$$u^3(\tau) = \frac{1}{\sqrt{A + B\, e^{-2\,\tau_0\,(\lambda_E^2+\lambda_B^2)\,\tau}}}\,\gamma_0\, c\,\left[\mathrm{sh}(\omega_E\,\tau) + \beta_0^z\,\mathrm{ch}(\omega_E\,\tau)\right]$$

$$u^1(\tau) = \frac{1}{\sqrt{B + A\, e^{+2\,\tau_0\,(\lambda_E^2+\lambda_B^2)\,\tau}}}\,\gamma_0\, c\,\left[\beta_0^x\,\cos(\omega_B\,\tau) + \beta_0^y\,\sin(\omega_B\,\tau)\right]$$

$$u^2(\tau) = \frac{1}{\sqrt{B + A\, e^{+2\,\tau_0\,(\lambda_E^2+\lambda_B^2)\,\tau}}}\,\gamma_0\, c\,\left[-\beta_0^x\,\sin(\omega_B\,\tau) + \beta_0^y\,\cos(\omega_B\,\tau)\right].$$

λ_E^2 and λ_B^2 eigenvalues of $F_i^{\ k}$ and A, B are the initial conditions for the 4-velocity.

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Numerical implementation of the solutions

- 2 versions of the pusher from (Hadad et al., 2010)
  1. full time-dependent solutions.
  2. time-independent solutions assuming E and B constant.
- valid only for null-like fields i.e. vanishing electromagnetic invariants :

$$\vec{E} \cdot \vec{B} = 0 \quad ; \quad E^2 - c^2 B^2 = 0.$$

- many tests of the code performed for analytical situations (Pétri, 2021).
  - linear, circular, elliptical polarized wave.
  - particle starting at rest or with initial ultra-relativistic speed.
  - different initial phase of the wave.
  - with and without radiation reaction.
  - $\Rightarrow$ constant E and B algorithm as good as time-dependent solver.
- we applied it to astrophysical situations for spherical waves with amplitude decreasing like $E, B \propto 1/r$ comparing the two versions for any polarization : circular, linear, elliptical.
- the strength parameter of the wave decreases with radius

$$a(r) = a_0 \frac{r_L}{r}$$

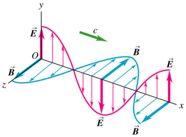

Fig. – Plane wave.

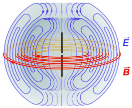

Fig. – Spherical wave.

Problem set up

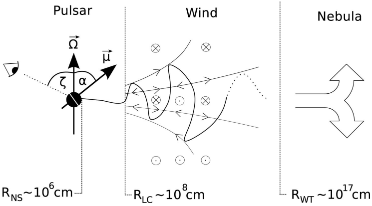

Fig. – Set up geometry.

- injection of particles at the light-cylinder or beyond at different colatitudes.
- evolving in circular, linear or elliptical spherical waves.
- with and without initial speed.
- with and without radiation reaction.

Particle starting at rest at light-cylinder

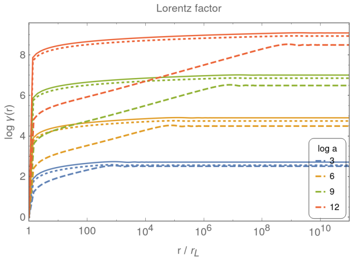

Fig. – Evolution of the Lorentz factor for a circularly, elliptically and linearly polarized wave respectively in solid, dotted and dashed line.

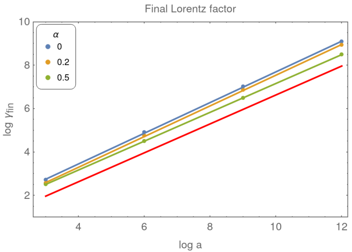

Fig. – Final Lorentz factor for a circularly, elliptically and linearly polarized wave. The law in Eq. (3) *is shown in red solid line.*

- In a spherical wave, the max Lorentz factor does not scale with $1 + a^2$ but only as

$$\gamma_{\rm fin} \approx 2\,(a/\pi)^{2/3} \tag{3}$$

- for linear polarization, the Lorentz factor increase is much slower and leads to weaker acceleration.
- radiation reaction does not impact these results.

Particle starting at rest at large distances

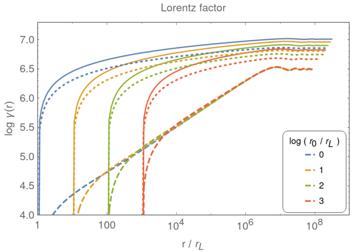

Fig. – Evolution of the Lorentz factor.

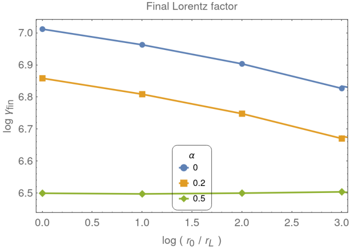

Fig. – Maximum Lorentz factor.

- the larger the initial position, the weaker the final Lorentz factor.
- linear polarization is always less efficient than circular or elliptical.
- radiation reaction does not impact these results.

Particle injected at relativistic speed at the light-cylinder

Leptons can be created close to the light-cylinder moving at relativistic speed and entering the wave at high Lorentz factor.

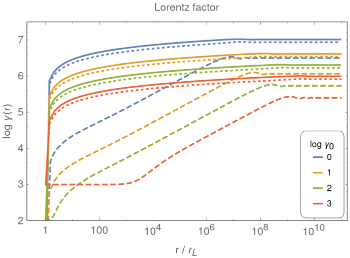

Fig. – Evolution of the Lorentz factor for a linearly polarized wave ($\alpha = 0.5$) with $a = 10^9$, initial Lorentz factor $\log \gamma_0 = \{0, 1, 2, 3\}$ and varying initial phases, $\xi_0 = 0$ in dashed lines, $\xi_0 = \pi/4$ in dotted lines and $\xi_0 = \pi/2$ in solid lines.

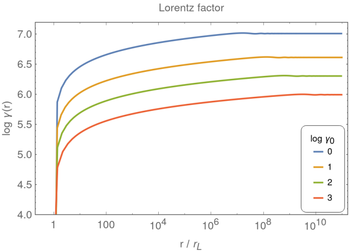

Fig. – Evolution of the Lorentz factor for a circularly polarized wave with $a = 10^9$, initial Lorentz factor $\log \gamma_0 = \{0, 1, 2, 3\}$. The curves are insensitive to the initial phase ξ_0.

- particles injected at high Lorentz factor in the node of a linear polarized wave are not easily accelerated.
- the initial phase of linear polarization strongly impacts on the time evolution.
- circular polarization is insensitive to initial phase (rotational symmetry).
- In all cases, radiation reaction negligible far from the star
- but becomes dominant close to its surface.

Head on collision

A particle from infinity enters the wave at ultra-relativistic speed and is reflected by the wave.

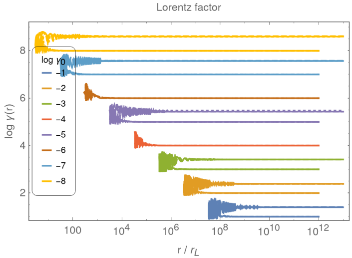

Fig. – Evolution of the Lorentz factor for a linearly polarized wave with $a = 10^9$.

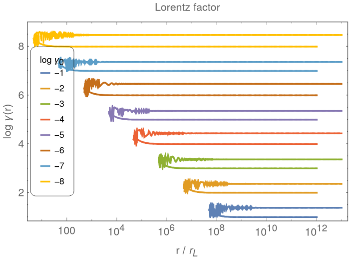

Fig. – Same as Fig. 14 but for circular polarization.

- final Lorentz factor after bounce $\gamma_{\rm fin} = \gamma_0 + 3\sqrt{\gamma_0^2 - 1} \approx 4\,\gamma_0$.
- minimal approach distance $\log(r_{\rm min}/r_{\rm L}) \approx (8.7)_{\rm circ}/(8.5)_{\rm lin} - \log\gamma_0$
- from simple physical arguments we expect

$$\gamma_0 \approx \frac{\omega_{\rm B}}{\omega} = a = a_0\,\frac{r_{\rm L}}{r_{\rm min}}.$$

$\Rightarrow$ in good agreement with the simulations.

Numerical scheme for constant fields

- assume that all quantities ($\vec{u}, \vec{x}, \vec{E}, \vec{B}$) are known at the time step τ^n.
- analytical solution for u^i known according to $\vec{u}^{n+1} = L(\Delta\tau, \vec{u}_0, \vec{E}, \vec{B})$.
- update in particle position performed by the velocity-Verlet algorithm

$$\begin{aligned}
\vec{u}^{n+1/2} &= \vec{L}(\Delta\tau/2, \vec{u}^n, \vec{E}(\vec{x}^n), \vec{B}(\vec{x}^n)) \\
\vec{x}^{n+1} &= \vec{x}^n + \vec{u}^{n+1/2}\,\Delta\tau \\
\vec{u}^{n+1} &= \vec{L}(\Delta\tau/2, \vec{u}^{n+1/2}, \vec{E}(\vec{x}^{n+1}), \vec{B}(\vec{x}^{n+1})).
\end{aligned}$$

- algorithm second order in proper time.
- extensively tested in electromagnetic field configurations with known analytical solutions.

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Particle tracking around neutron stars

- inject charged particles and let them evolve under LLR
  - electron
  - proton
  - iron.
- in the electromagnetic field of a neutron star
  - millisecond pulsar

    $$B \sim 10^5 \ T \qquad and \qquad a \sim 10^{13}$$

  - normal pulsar

    $$B \sim 10^8 \ T \qquad and \qquad a \sim 10^{18}$$

  - magnetar

    $$B \sim 10^{11} \ T \qquad and \qquad a \sim 10^{21}$$

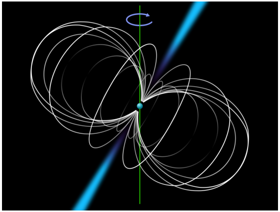


Fig. – Particle injection in the dipole field.

- three kind of motion
  - trapped
  - crashed
  - escaped.

(Pétri, 2022) http ://arxiv.org/abs/2207.00624

Particles escaping and crashing around neutron stars

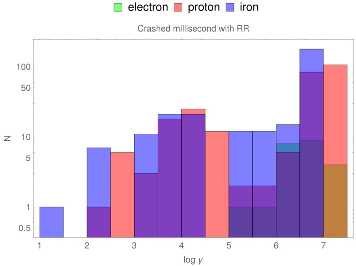


Fig. – Final Lorentz factor for electrons, protons and irons with RR.

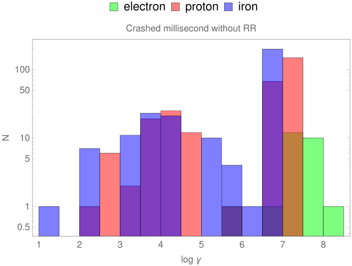


Fig. – Final Lorentz factor for electrons, protons and irons without RR.

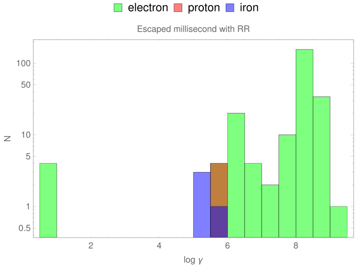


Fig. – Final Lorentz factor for electrons, protons and irons with RR.

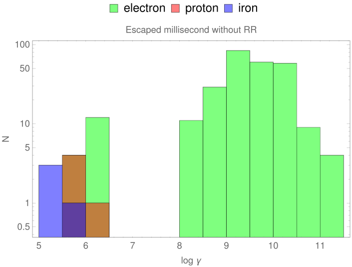


Fig. – Final Lorentz factor for electrons, protons and irons without RR.

Particles trapped around neutron stars

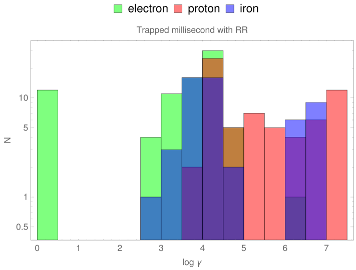
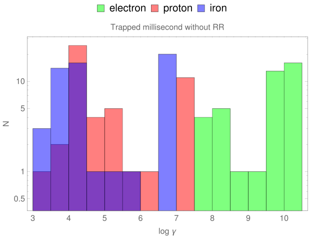


Fig. – Final Lorentz factor for electrons, protons and irons with RR.



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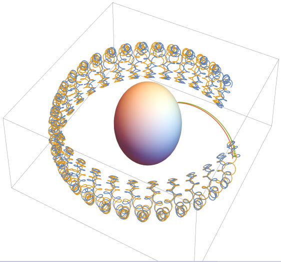


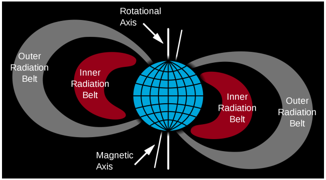


Fig. – Schematic 2D view of Earth Van Allen radiation belt.

An alternative approach

- solving LLR is very time consuming computationally.
- when the Lorentz force is exactly balanced by radiation reaction

$$\mathbf{E} + \mathbf{v} \wedge \mathbf{B} \approx \pm K\, \mathbf{v}$$

- in a strong radiation reaction limit (RRL) regime in can be approximated by

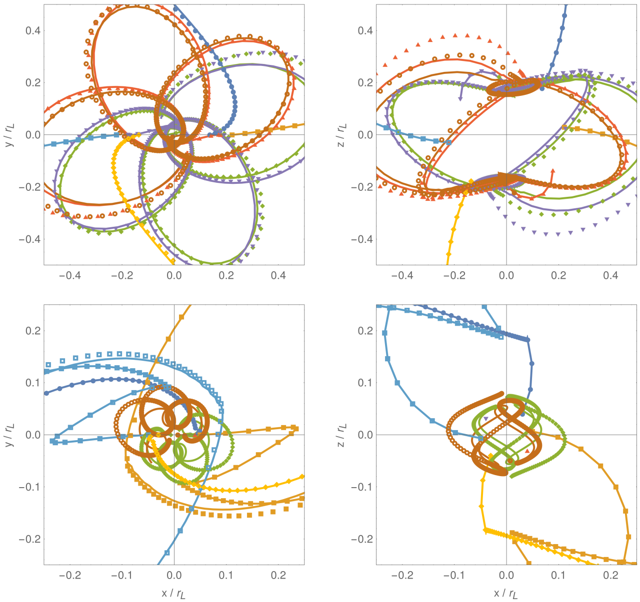
$$\mathbf{v}_{\pm} = \frac{\mathbf{E} \wedge \mathbf{B} \pm (E_0\, \mathbf{E}/c + c\, B_0\, \mathbf{B})}{E_0^2/c^2 + B^2}$$

  assuming particles moving at the speed of light $\|\mathbf{v}_{\pm}\| = c$.
- invariants $\mathbf{E}^2 - c^2\, \mathbf{B}^2 = E_0^2 - c^2\, B_0^2$ and $\mathbf{E} \cdot \mathbf{B} = E_0\, B_0$ with $K = E_0/c \geq 0$.
- meaning = there exist a frame where $\mathbf{E}'$ and $\mathbf{B}'$ are parallel and the particle moves at the speed of light along the common direction.

$\Rightarrow$ not possible when both invariants vanish (null like fields, $E_0 = B_0 = 0$).

- RRL gives accurate results without integrating the full equation of motion in time !

Comparison of particle trajectories in LLR and RRL approximations



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- neutron stars are very efficient particle accelerators (UHE cosmic rays ?).
- realistic physical parameters required to avoid artificial down-scaling.
- algorithm to solve analytically for the equation of motion in LLR approximation.
- RRL regime gives very similar results at low computational cost.
- applied it to a rotating neutron star with particle Lorentz factors up to 10^{12}.
- efficiency of radiation damping depending on stellar magnetic fields strength, period and damping parameter.

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